# LECTURE NOTES

# ON

# "ENGINEERING MATHEMATICS-III"


**Prepared By:**

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# Th1. ENGINEERING MATHEMATICS – III

**(COMMON TO ELECT,ETC, AE & I and other Allied branches of Electrical and ETC )**

| Name of the Course: Diploma in Electrical Engineering | | | |
|---|---|---|---|
| Course code: | | Semester | 3rd |
| Total Period: | 60 | Examination : | 3 hrs |
| Theory periods: | 4P / week | Internal Assessment: | 20 |
| Maximum marks: | 100 | End Semester Examination :: | 80 |

## A. RATIONALE:

The subject engineering mathematics-III is a common paper for engineering branches. This subject includes complex numbers, Matrices, Laplace Transforms, Fourier series, Differential equations and Numerical Methods etc for solution of engineering problems.

## B. OBJECTIVE:

On completion of study of Engineering Mathematics-III, the students will be able to:

1. Apply complex number concept in electricity , Quadratic equation , Imaginary numbers in signal processing, Radar & even biology ( Brain Waves )
2. Apply Matrices in Engineering fields such as Electrical Circuits and Linear programming.
3. Transform Engineering problems to mathematical models with the help of differential equations and familiarize with the methods of solving by Analytical methods, Transform method and operator method and Numerical methods.
4. Solve algebraic equations by iterative Methods easily programmable in computers.
5. Analysis data and develop interpolating polynomials through method of differences

**C. Topic wise distribution of periods:**

| Sl. No. | Topics | Period |
|---|---|---|
| 1 | Complex Numbers | 06 |
| 2 | Matrices | 04 |
| 3 | Differential Equations | 10 |
| 4 | Laplace transforms | 12 |
| 5 | Fourier Series | 12 |
| 6 | Numerical Methods | 04 |
| 7 | Finite difference & interpolation | 12 |
| | **Total:** | **60** |

## D. COURSE CONTENTS

**1. Complex Numbers**

1.1 Real and Imaginary numbers.
1.2 Complex numbers, conjugate complex numbers, Modulus and Amplitude of a complex number.
1.3 Geometrical Representation of Complex Numbers.
1.4 Properties of Complex Numbers.
1.5 Determination of three cube roots of unity and their properties.

1.6 De Moivre's theorem
1.7 Solve problems on 1·1 - 1·6

**2. Matrices**

2.1. Define rank of a matrix.
2.2. Perform elementary row transformations to determine the rank of a matrix.
2.3. State Rouche's theorem for consistency of a system of linear equations in $n$ unknowns.
2.4. Solve equations in three unknowns testing consistency.
2.5. Solve problems on 2.1 – 2.4

**3. Linear Differential Equations**

3.1. Define Homogeneous and Non – Homogeneous Linear Differential Equations with constant coefficients with examples.
3.2. Find general solution of linear Differential Equations in terms of C.F. and P.I.
3.3. Derive rules for finding C.F. And P.I. in terms of operator D, excluding $\frac{1}{f(D)}x^n$.
3.4. Define partial differential equation (P.D.E) .
3.5. Form partial differential equations by eliminating arbitrary constants and arbitrary functions.
3.6. Solve partial differential equations of the form Pp + Qq = R
3.7. Solve problems on 3.1- 3.6

**4. Laplace Transforms**

4.1. Define Gamma function and $\Gamma(n+1) = n!$ and find $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ .
4.2. Define Laplace Transform of a function $f(t)$ and Inverse Laplace Transform .
4.3. Derive L.T. of standard functions and explain existence conditions of L.T.
4.4. Explain linear, shifting property of L.T.
4.5. Formulate L.T. of derivatives, integrals, multiplication by $t^n$ and division by $t$.
4.6. Derive formulae of inverse L.T. and explain method of partial fractions .
4.7. solve problem on 4.1- 4.6

**5. Fourier Series**

5.1. Define periodic functions.
5.2. State Dirichlet's condition for the Fourier expansion of a function and it's convergence
5.3. Express periodic function $f(x)$ satisfying Dirichlet's conditions as a Fourier series.
5.4. State Euler's formulae.
5.5. Define Even and Odd functions and find Fourier Series in ( $0 \leq x \leq 2\pi \; and -\pi \leq x \leq \pi$ ).
5.6. Obtain F.S of continuous functions and functions having points of discontinuity in ( $0 \leq x \leq 2\pi \; and -\pi \leq x \leq \pi$ )
5.7. Solve problems on 5.1 – 5.6

**6. Numerical Methods**

6.1. Appraise limitation of analytical methods of solution of Algebraic Equations.
6.2. Derive Iterative formula for finding the solutions of Algebraic Equations by :

- 6.2.1. Bisection method
- 6.2.2. Newton- Raphson method
- 6.3. solve problems on 6.2

**7. Finite difference and interpolation**

- 7.1. Explain finite difference and form table of forward and backward difference.
- 7.2. Define shift Operator $(E)$ and establish relation between $E$ & difference operator$(\Delta)$.
- 7.3. Derive Newton's forward and backward interpolation formula for equal intervals.
- 7.4. State Lagrange's interpretation formula for unequal intervals.
- 7.5. Explain numerical integration and state:
  - 7.5.1. Newton's Cote's formula.
  - 7.5.2. Trapezoidal rule.
  - 7.5.3. Simpson's 1/3rd rule
- 7.6. Solve problems on 7.1- 7.5

**Syllabus to be covered up to I.A**.
Chapter: 1,2,3 and 4

**Learning Resources:**

| Sl.No | Title of the Book | Name of Authors | Name of Publisher |
|---|---|---|---|
| 1. | Higher engineering mathematics | Dr B.S. Grewal | khanna publishers |
| 2. | Elements of mathematics Vol-1 | Odisha state bureau of text book preparation and production | |
| 3. | Text Book of Engineering Mathematics-I | C.R Mallick | Kalayani publication |
| 4. | Text Book of engineering mathematics-III | C.R Mallick | Kalayani publication |

# COMPLEX NUMBERS

**ALGEBRA OF COMPLEX NUMBERS:**

Definition – Real and Imaginary parts, Conjugates, Modulus and amplitude form, Polar form of a complex number, multiplication and division of complex numbers (geometrical proof not needed) – Simple Problems. Argan Diagram – Collinear points, four points forming square, rectangle, rhombus and parallelogram only. Simple problems.

DE MOIVRE’S THEOREM

Demoivre’s Theorem (Statement only) – related simple problems.

ROOTS OF COMPLEX NUMBERS

Finding the nth roots of unity - solving equation of the form $x^n \pm 1 = 0$ where $n \leq 7$. Simple problems.

## ALGEBRA OF COMPLEX NUMBERS

### Introduction:

Let us consider the quadratic equation $ax^2 + bx + c = 0$. The solution of this equation is given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is meaningful only when $b^2 - 4ac > 0$. Because the square of a real number is always positive and it cannot be negative. If it is negative, then the solution for the equation extends the real number system to a new kind of number system that allows the square root of negative numbers. The square root of – 1 is denoted by the symbol i, called the imaginary unit, which was first introduced in mathematics by the famous Swiss mathematician, Leonhard Euler in 1748. Thus for any two real numbers a and b, we can form a new number a + ib is called a **complex number**. The set of all complex numbers denoted by C and the nomenclature of a complex number was introduced by a German mathematician C.F. Gauss.

### Definition: Complex Number

A number which is of the form a + ib where $a, b \in R$ and $i^2 = -1$ is called a complex number and it is denoted by z. If z = a + ib then *a* is called the real part of z and *b* is called the imaginary part of z and are denoted by Re(z) and Im(z).

For example, if z = 3 + 4i then Re (z) = 3 and Im (z) = 4.

***Note:***

In the complex number z = a + ib we have,

(i) If a = 0 then z is purely imaginary

(ii) If b = 0 then z is purely real.

(iii) z = a + ib = (a, b) any complex number can be expressed as an ordered pair.

### Conjugate of a complex number:

If z = a + ib then the conjugate of z is defined by a – ib and it is denoted by $\bar{z}$. Thus, if z = a + ib then $\bar{z} = a - ib$.

**Results:**

(i) $\overline{\overline{z}} = z$

(ii) $a = \text{Re}(z) = \dfrac{z+\overline{z}}{2}$ & $b = I_m(z) = \dfrac{z-\overline{z}}{2}$

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(iv) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(v) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ where $z_2 \neq 0$

(vii) $\overline{z^n} = (\overline{z})^n$

## Algebra of complex numbers:

***(i) Addition of two complex numbers:***

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their sum is defined as

$z_1 + z_2 = a + ib + c + id = (a + c) + i\,(b + d) \in C$

$z + \overline{z} = 2a$ Real number.

***(ii) Difference of two complex numbers:***

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their difference is defined as

$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i\,(b - d) \in C$

$z - \overline{z} = 2ib$ Imaginary number.

***(iii) Multiplication of two compelx numbers:***

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers then their product is defined as,

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + i\,ad + i\,bc + i^2 bd \\ &= (ac - bd) + i\,(ad + bc) \in C \end{aligned}$$

$z\,\overline{z} = (a + ib)(a - ib) = a^2 + b^2$

***(iv) Division of two complex numbers:***

Let $z_1 = a + ib$ and $z_2 = c + id \neq 0$ be any two complex numbers then their quotient is defined as

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]$$

## Modulus of a complex number:

If $z = a + ib$ is a complex number then the modulus (or) absolute value of z is defined as $\sqrt{a^2 + b^2}$ and is denoted by $|z|$. Thus, if $z = a + ib$ then $|z| = \sqrt{a^2 + b^2}$ .

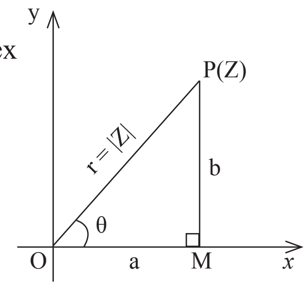
***Note:***

(i) $|\overline{z}| = |z| = \sqrt{a^2 + b^2}$

(ii) $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

(iii) $\text{Re}(z) \leq |z|$ and $\text{Im}(z) \leq |z|$

## Polar form of a Complex Number:




Let (r, θ) be the Polar co-ordinates of the point P representing the complex number z = a + ib. Then from the fig. we get,

$$\cos\theta = \frac{OM}{OP} = \frac{a}{r} \text{ and } \sin\theta = \frac{PM}{OP} = \frac{b}{r}$$

$\Rightarrow$ a = r cos θ and b = r sin θ

where $r = \sqrt{a^2 + b^2} = |a + ib|$ is called the **modulus** of z = a + ib.

Also, $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ is called the **amplitude** or **argument** of z = a + ib and denoted by amp(z) or arg(z) and is measured as the angle in positive sense. Thus, arg(z) = θ = $\tan^{-1}\left(\frac{b}{a}\right)$.

Hence z = a + ib = r (cos θ + i sin θ) is called the Polar form or the modulus amplitude form of the complex number.

## Theorems of Complex numbers:

1) The product of two complex numbers is a complex number whose modulus is the product of their modulii and whose amplitude is the sum of their amplitudes

   i.e, $|z_1 z_2| = |z_1||z_2|$

   and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

2) The quotient of two complex numbers is a complex number whose modulus is the quotient of their modulii and whose amplitude is the difference of their amplitudes.

   i.e $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ wehre $Z_2 \neq 0$ and arg $\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ .

## Euler's formula:

The symbol $e^{i\theta}$ is defined by $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula.

If z ≠ 0 then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. This is called the exponential form of the complex number z.

***Note:*** If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$ .

## Multiplication and Division of complex numbers (Geometrical proof not needed)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be any two complex numbers in Polar form then their product is given by

$$z_1z_2 = r_1r_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Also the division of the above two complex numbers is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \text{ where } z_2 \neq 0.$$

## WORKED EXAMPLES

## PART – A

1. If $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$ find $z_1 + z_2$.

***Solution:***

Given: $z_1 = 2 + 3i$ & $z_2 = 4 - 5i$

$$\begin{aligned} z_1 + z_2 &= (2 + 3i) + (4 - 5i) \\ &= 2 + 3i + 4 - 5i \\ &= (2 + 4) + (3i - 5i) \end{aligned}$$

$$\Rightarrow \boxed{z_1 + z_2 = 6 - 2i}$$

2. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$ find the value of $2z_1 - 3z_2$.

***Solution:***

Given: $z_1 = 3 - 4i$ & $z_2 = -2 + 3i$

$$\begin{aligned} 2z_1 - 3z_2 &= 2(3 - 4i) - 3(-2 + 3i) \\ &= 6 - 8i + 6 - 9i \end{aligned}$$

$$\Rightarrow \boxed{2z_1 - 3z_2 = 12 - 17i}$$

3. Express: $(3 + 2i)(4 + 2i)$ in $a + ib$ form.

***Solution:***

$$\begin{aligned} (3 + 2i)(4 + 2i) &= 12 + 6i + 8i + 4i^2 \\ &= 12 + 14i - 4 \\ &= 8 + 14i = a + ib \text{ form.} \end{aligned}$$

4. Find the real and imaginary parts of $\frac{1}{3 + 2i}$.

***Solution:***

$$\text{Let } z = \frac{1}{3+2i} = \frac{1}{3+2i} \times \frac{3-2i}{3-2i}$$

$$= \frac{3-2i}{(3)^2 - (2i)^2}$$

$$= \frac{3-2i}{9+4}$$

$$= \frac{3-2i}{13}$$

$$\Rightarrow \boxed{z = \frac{3}{13} - \frac{2i}{13}}$$

$$\therefore \text{Re}(z) = \frac{3}{13} \ \& \ \text{Im}(z) = \frac{-2}{13}$$

5. Find the conjugate of $\frac{1}{1+i}$.

***Solution:***

$$\text{Let } z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{(1)^2 - (i)^2}$$

$$= \frac{1-i}{1+1}$$

$$= \frac{1-i}{2}$$

$$\Rightarrow \boxed{z = \frac{1}{2} - \frac{i}{2}}$$

$$\therefore \text{Conjugate} : \bar{z} = \frac{1}{2} + \frac{i}{2}$$

6. Find the modulus and amplitude of 1 + i.

***Solution:***

Let z = 1 + i

Here a = 1 & b = 1

$$\text{Modulus} : |z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\text{and amp}(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$$

$$\Rightarrow \theta = 45^{o}$$

## PART –B

1. Find the real and imaginary parts of $\frac{4+5i}{3-2i}$.

***Solution:***

$$\text{Let } z = \frac{4+5i}{3-2i} = \frac{4+5i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{12+8i+15i+10i^2}{(3)^2 - (2i)^2}$$

$$= \frac{12+23i-10}{9+4}$$

$$= \frac{2+23i}{13}$$

$$z = \frac{2}{13} + \frac{23i}{13}$$

$$\therefore \text{Re}(z) = \frac{2}{13} \quad \& \quad \text{Im}(z) = \frac{23}{13}$$

2. Express the complex number $\frac{1}{3-2i}+\frac{1}{2-3i}$ in a + ib form.

***Solution:***

$$\text{Let } z=\frac{1}{3-2i}+\frac{1}{2-3i}$$

$$=\frac{1}{3-2i}\times\frac{3+2i}{3+2i}+\frac{1}{2-3i}+\frac{2+3i}{2+3i}$$

$$=\frac{3+2i}{3^2+2^2}+\frac{2+3i}{2^2+3^2}$$

$$=\frac{3+2i+2+3i}{13}$$

$$=\frac{5+5i}{13}$$

$$z=\frac{5}{13}+\frac{5}{13}i=a+ib\text{ form}$$

3. Find the modulus and argument of the complex number $\frac{1-i}{1+i}$.

***Solution:***

$$\text{Let } z=\frac{1-i}{1+i}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}$$

$$=\frac{1-i-i+i^2}{(1)^2-(i)^2}$$

$$=\frac{1-2i-1}{1+1}$$

$$=\frac{-2i}{2}$$

$$z=-i\quad\text{where } a=0\ \&\ b=-1$$

$$\text{Modulus}:|z|=\sqrt{a^2+b^2}=\sqrt{(0)^2+(-1)^2}=\sqrt{1}=1$$

$$\text{Argument}:\tan\theta=\frac{b}{a}=\frac{-1}{0}=\infty$$

$$\theta=\tan^{-1}(\infty)=90^{o}$$

The complex number – i = (0, – 1) lies IIIrd Quadrant.

Hence amplitude = $180^{o}+90^{o}=270^{o}$.

## PART – C

1. Find the real and imaginary parts of the complex number $\frac{(1+i)\,(2-i)}{1+3i}$.

***Solution:***

$$\text{Let } z=\frac{(1+i)\,(2-i)}{1+3i}$$

$$=\frac{2-i+2i-i^2}{1+3i}$$

$$=\frac{2+i+1}{1+3i}$$

$$= \frac{3+i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{3-9i+i-3i^2}{(1)^2-(3i)^2}$$

$$= \frac{3-8i+3}{1+9}$$

$$= \frac{6-8i}{10}$$

$$= \frac{3-4i}{5}$$

$$z = \frac{3}{5} - \frac{4i}{5} = a + ib \text{ form.}$$

$$\therefore \text{Re}(z) = \frac{3}{5} \ \& \ \text{Im}(z) = -\frac{4}{5}$$

2. Express the complex number $\frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$ in a + ib form.

***Solution:***

$$\text{Let } z = \frac{i-4}{3-2i} + \frac{4i+1}{2-3i}$$

$$= \frac{i-4}{3-2i} \times \frac{3+2i}{3+2i} + \frac{4i+1}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{3i-12+2i^2-8i}{3^2+2^2} + \frac{8i+2+12i^2+3i}{2^2+3^2}$$

$$= \frac{-5i-14}{13} + \frac{11i-10}{13}$$

$$= \frac{6i-24}{13}$$

$$= \frac{-24+6i}{13} = \frac{-24}{13} + \frac{6i}{13} = a + ib \text{ form}$$

3. Find the modulus and amplitude of $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$.

***Solution:***

$$\text{Let } z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$$

$$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$$

$$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{(\sqrt{3})^2-(2i)^2}$$

$$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4}$$

$$= \frac{7\sqrt{3}+7i}{7}$$

$$= \frac{7(\sqrt{3}+i)}{7}$$

$$z = \sqrt{3} + i = a + ib \text{ form}$$

Here $a = \sqrt{3}$ & $b = 1$

$\therefore$ Modulus $: |z| = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (1)^2} = \sqrt{3+1} = \quad 4 = 2$

Amplitude $: \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^{\circ}$

4. Find the modulus and argument of the complex number $\frac{5-i}{2-3i}$.

***Solution:***

Let $z = \frac{5-i}{2-3i}$

$= \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{10+15i-2i-3i^2}{(2)^2-(3i)^2}$

$= \frac{10+13i+3}{4+9}$

$= \frac{13+13i}{13}$

$= \frac{13(1+i)}{13}$

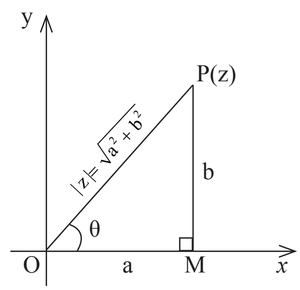
$z = 1 + i = a + ib$ form

Here $a = 1$ & $b = 1$

Modulus $: |z| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

Amplitude $: \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$

$\Rightarrow \theta = 45^{\circ}$




## Argand Diagram

Every complex number a + ib can be considered as an ordered pair (a, b) of real numbers, we can represent such number by a point in xy-plane called the complex plane and such a representation is also known as the argand diagram. The complex number z = a + ib represented by P(z) then the distance between z and the origin is the modulus. i.e $|z| = \sqrt{a^2 + b^2}$

Here the set of real numbers $(x, 0)$ corresponds to the x-axis called real axis and the set of Imaginary numbers (0, y) correponds to the y-axis called the imaginary axis.

***Result:***

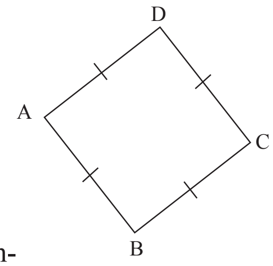
The distance between the two complex numbers $z_1$ and $z_2$ is $|z_1 - z_2|$. Thus, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .

## Collinear Points:

If A, B and C are any three points representing the complex numbers $x_1 + iy_1$, $x_2 + iy_2$ and $x_3 + iy_3$ respectively, are collinear then the required condition is, the area of $\Delta$ABC is zero.

i.e. $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

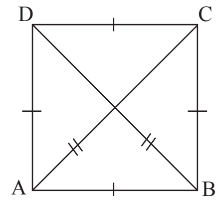
i.e $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$




i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

## Condition for square:

If A, B, C and D are any four complex numbers representing the vertices of a square then the required condition is




(i) AB = BC = CD = DA

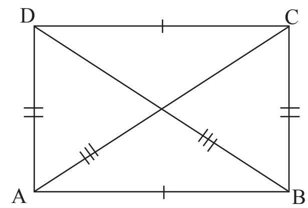
i.e four sides are equal.

(ii) AC = BD

i.e the diagonals are equal

## Condition for rectangle

If A, B, C and D are any four complex numbers represents the vertices of a rectangle then the required condition is,




(i) AB = CD and BC = DA.

i.e Opposite sides are equal.

(ii) AC = BD

i.e the diagonals are equal.

## Condition for rhombus

If A, B, C and D are any four complex numbers represents the vertices of a rhombus then the required condition is

AB = BC = CD = DA

i.e four sides are equal.

## Condition for Parallelogram

If A, B, C and D are any four complex numbers represents the vertices of a parallelogram then the required condition is either,

mid-point of the diagonal AC = mid.point of the diagonal BD.

(or) The length of the opposite sides are equal. i.e AB = CD & BC = DA.

***Note:***

(i) To find length of the sides and diagonals of square, rectangle, rhombus and parallelogram apply the distance formula.

i.e distance: AB = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ etc.

(ii) To find the mid.point of the diagonals of the parallelogram apply middle point formula,

$$(x, y) = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right]$$

## WORKED EXAMPLES

### PART – A

1. Find the distance between the complex numbers 2 + i and 1 –2i.

***Solution:***

Let $z_1 = 2 + i$ & $z_2 = 1 - 2i$

$$\therefore \text{Distance}: z_1z_2 = \sqrt{(2-1)^2 + (1+2)^2}$$

$$= \sqrt{(1)^2 + (3)^2}$$

$$= \sqrt{1+9}$$

$$\Rightarrow \boxed{z_1z_2 = \sqrt{10}}$$

### PART – B

1. Show that the complex numbers 1 + 2i, 3 – 2i and 6 – 8i are collinear.

***Solution:***

Given complex numbers are

A : 1 + 2i = (1, 2)

B : 3 – 2i = (3, – 2)

& C : 6 – 8i = (6, – 8)

Condition for collinear is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\text{LHS}: \begin{vmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 6 & -8 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -8 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 6 & -8 \end{vmatrix}$$

$$= 1(-2+8) - 2(3-6) + 1(-24+12)$$

$$= 1(6) - 2(-3) + 1(-12)$$

$$= 6 + 6 - 12$$

$$= 12 - 12$$

$$= 0$$

∴ The given complex numebrs are collinear.

2. Show that the complex numbers – 1 + i, 3 + 2i, 2 + 2i and – 2 + i form a parallelogram.

***Solution:***

Let the given complex numbers are

A : – 1 + i = (– 1, 1)

B : 3 + 2i = (3, 2)

C : 2 + 2i = (2, 2)

D : – 2 + i = (– 2, 1)

$$\text{Mid-point of AC} = \left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right] = \left[\frac{-1+2}{2}, \frac{1+2}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right] \text{............(1)}$$

$$\text{Mid-point of BD} = \left[\frac{3-2}{2}, \frac{2+1}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right] \text{............(2)}$$

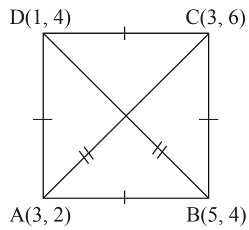
From (1) & (2), Mid-point of AC = Mid-point of BD.

⇒ Given complex numbers form a parallelogram.

## PART – C

1. Prove that the points representing the complex numbers 3 + 2i, 5 + 4i, 3 + 6i and 1 + 4i form a square.

***Solution:***




Let the given complex number be

A : 3 + 2i = (3, 2)

B : 5 + 4i = (5, 4)

C : 3 + 6i = (3, 6)

& D : 1 + 4i = (1, 4)

Now,

$$AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-5)^2+(2-4)^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$BC = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(5-3)^2+(4-6)^2} = \sqrt{(2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$CD = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-1)^2+(6-4)^2} = \sqrt{(2)^2+(2)^2} = \sqrt{4+4} = \sqrt{8}$$

and $$DA = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(1-3)^2+(4-2)^2} = \sqrt{(-2)^2+(2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$\text{Also, } AC = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(3-3)^2+(2-6)^2} = \sqrt{(0)^2+(-4)^2} = \sqrt{16} = 4$$

$$BD = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = \sqrt{(5-1)^2+(4-4)^2} = \sqrt{(4)^2+(0)^2} = \sqrt{16} = 4$$

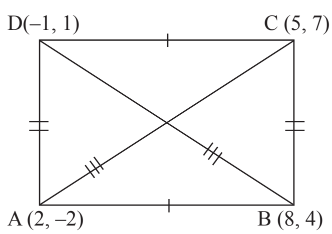
Here the sides AB = BC = CD = DA = $\sqrt{8}$

& diagonals AC = BD = 4

∴ The given complex numbers form a square.

2. Prove that the points representing the complex numbers 2 –2i, 8 + 4i, 5 + 7i and – 1 + i form a rectangle.

***Solution:***




Let the given complex numbers be,

A : 2 –2i = (2, – 2)

B : 8 + 4i = (8, 4)

C : 5 + 7i = (5, 7)

& D : – 1 + i = (– 1, 1)

Now, $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2-8)^2 + (-2-4)^2} = \sqrt{(-6)^2 + (-6)^2} = \sqrt{36+36} = \sqrt{72}$

Now, $BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(8-5)^2 + (4-7)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(5+1)^2 + (7-1)^2} = \sqrt{(6)^2 + (6)^2} = \sqrt{36+36} = \sqrt{72}$

& $DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-1-2)^2 + (1+2)^2} = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18}$

Also, $AC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2-5)^2 + (-2-7)^2} = \sqrt{(-3)^2 + (-9)^2} = \sqrt{9+81} = \sqrt{90}$

$BD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(8+1)^2 + (4-1)^2} = \sqrt{(9)^2 + (3)^2} = \sqrt{81+9} = \sqrt{90}$

Here the sides $AB = CD = \sqrt{72}$

$BC = DA = \sqrt{18}$

and diagonals $AC = BD = \sqrt{90}$

∴ The given complex numbers form a rectangle.

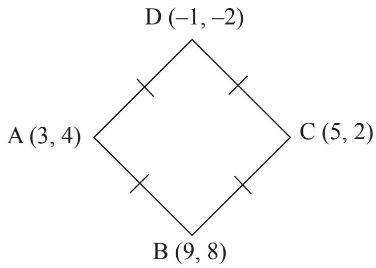
3. Prove that the points 3 + 4i, 9 + 8i, 5 + 2i and – 1 – 2i form a rhombus in the argand diagram.

***Solution:***

Let the given complex numbers be,

A : 3 + 4i = (3, 4) B : 9 + 8i = (9, 8)

C : 5 + 2i = (5, 2) & D : – 1 – 2i = (– 1, – 2)




$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3-9)^2 + (4-8)^2} = \sqrt{(-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52}$

$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(9-5)^2 + (8-2)^2} = \sqrt{(4)^2 + (6)^2} = \sqrt{16+36} = \sqrt{52}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(5+1)^2 + (2+2)^2} = \sqrt{(6)^2 + (4)^2} = \sqrt{36+16} = \sqrt{52}$

& $DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-1-3)^2 + (-2-4)^2} = \sqrt{(-4)^2 + (-6)^2} = \sqrt{16+36} = \sqrt{52}$

Here the sides, $AB = BC = CD = DA = \sqrt{52}$ .

∴ The given complex number form a rhombus.

# DE-MOIVRE'S THEOREM

## DeMoivre's Thoerem: (Statement only)

(i) If 'n' is an integer positive or negative then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(ii) If 'n' is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

***Results:***

1) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

2) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$

3) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

4) $\dfrac{1}{\cos\theta - i\sin\theta} = (\cos\theta - i\sin\theta)^{-1} = \cos\theta + i\sin\theta$

5) $\sin\theta + i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)$

***Note:***

1) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

2) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

3) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n)$

### WORKED EXAMPLES

### PART – A

1. If $z = \cos 30^\circ + i\sin 30^\circ$ what is the value of $z^3$.

***Solution:***

$$z^3 = [\cos 30^\circ + i\sin 30^\circ]^3$$
$$= \cos 90^\circ + i\sin 90^\circ$$
$$= 0 + i(1) = i$$

2. If $z = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$ what is the value of $z^8$.

***Solution:***

$$z^8 = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^8$$
$$= \cos 8\left(\frac{\pi}{2}\right) + i\sin 8\left(\frac{\pi}{2}\right)$$
$$= \cos 4\pi + i\sin 4\pi$$
$$= 1 + i(0) = 1$$

3. If $z = \cos 45^\circ - i\sin 45^\circ$ what is the value of $\dfrac{1}{z}$.

***Solution:***

$$\frac{1}{z} = z^{-1}$$

$= [\cos 45 - i \sin 45^o]^{-1}$

$= \cos 45^o + i \sin 45^o$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

4. If $\frac{1}{z}$ = cos 60° + i sin 60° what is the value of z.

***Solution:***

$$z = \frac{1}{\frac{1}{z}}$$

$$= \frac{1}{\cos 60^o + i \sin 60^o} = (\cos 60^o + i \sin 60^o)^{-1}$$

$$= \cos 60^o - i \sin 60^o$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

5. Find the value of $\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i\sin\theta}$.

***Solution:***

$$\frac{\cos 3\theta + i \sin 3\theta}{\cos\theta - i\sin\theta} = (\cos 3\theta + i\sin 3\theta)(\cos\theta - i\sin\theta)^{-1}$$

$$= (\cos 3\theta + i\sin 3\theta)(\cos\theta + i\sin\theta)$$

$$= \cos(3\theta + \theta) + i\sin(3\theta + \theta)$$

$$= \cos 4\theta + i\sin 4\theta$$

6. Simplify: (cos 20° + i sin 20°) (cos 30° + i sin 30°) (cos 40° + i sin 40°)

***Solution:***

(cos 20° + i sin 20°) (cos 30° + i sin 30°) (cos 40° + i sin 40°)

= cos (20° + 30° + 40°) + i sin (20° + 30° + 40°)

= cos 90° + i sin 90°

= 0 + i (1) = i

7. If x = cos θ + i sin θ find $x + \frac{1}{x}$.

***Solution:***

Given: x = cos θ + i sin θ

$$\Rightarrow \quad \frac{1}{x} = \cos\theta - i\sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

8. If x = cos α + i sin α and y = cos β + i sin β find xy.

***Solution:***

$$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

9. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $\frac{a}{b}$.

***Solution:***

$$\frac{a}{b} = a(b)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)^{-1}$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$
$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

10. Find the product of $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

***Solution:***

$$3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 12\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$
$$= 12\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$
$$= 12\,[1+i(0)] = 12$$

## PART – B

1. If $x = \cos\theta + i\sin\theta$ find the value of $x^m + \frac{1}{x^m}$.

***Solution:***

Given: $x = \cos\theta + i\sin\theta$

$\Rightarrow x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

also, $\frac{1}{x^m} = (x^m)^{-1}$

$$= [\cos m\theta + i\sin m\theta]^{-1}$$

$$\Rightarrow \frac{1}{x^m} = \cos m\theta - i\sin m\theta$$

$$\therefore x^m + \frac{1}{x^m} = \cos m\theta + i\sin m\theta + \cos m\theta - i\sin m\theta = 2\cos m\theta$$

2. If $a = \cos\alpha + i\sin\alpha$ and $b = \cos\beta + i\sin\beta$ find $ab + \frac{1}{ab}$.

***Solution:***

Given: $a = \cos\alpha + i\sin\alpha$

& $b = \cos\beta + i\sin\beta$

$ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$\Rightarrow ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

and $\frac{1}{ab} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$

$$\therefore ab + \frac{1}{ab} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$
$$= 2\cos(\alpha+\beta)$$

3. Prove that $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)$.

***Solution:***

We have $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

$\& \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$

LHS: $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$

$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

4. If x = cos 3α + i sin 3α, y = cos 3β + i sin 3β find the value of $\sqrt[3]{xy}$.

***Solution:***

Given: x = cos 3α + i sin 3α

& y = cos 3β + i sin 3β

xy = (cos 3α + i sin 3α) (cos 3β + i sin 3β)

= cos (3α + 3β) + i sin (3α + 3β)

⇒ xy = cos 3 (α + β) + i sin 3 (α + β)

$\sqrt[3]{xy} = (xy)^{1/3}$

$= [\cos 3(\alpha+\beta) + i\sin 3(\alpha+\beta)]^{1/3}$

$= \cos\frac{1}{3}.3(\alpha+\beta) + i\sin\frac{1}{3}.3(\alpha+\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

5. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\frac{ab}{c}$.

***Solution:***

Given: a = cos α + i sin α

b = cos β + i sin β

& c = cos γ + i sin γ

$\therefore \frac{ab}{c} = ab(c)^{-1}$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)^{-1}$

$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma - i\sin\gamma)$

$\Rightarrow \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$

**PART – C**

1. Simplify: $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta + i\sin 4\theta)^{-3}}$

***Solution:***

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 4\theta + i\sin 4\theta)^{-3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{4\times -3}}{(\cos\theta + i\sin\theta)^{2\times 3} (\cos\theta + i\sin\theta)^{-3\times 4}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-6+12}$$

$$= (\cos\theta + i\sin\theta)^{0}$$

$$= \cos 0 + i\sin 0$$

$$= 1 + i(0) = 1$$

2. Simplify : $\dfrac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$ when $\theta = \dfrac{\pi}{9}$.

***Solution:***

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$$

$$= \frac{(\cos\theta + i\sin\theta)^{3\times 2} (\cos\theta + i\sin\theta)^{3\times -4}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{6} (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= (\cos\theta + i\sin\theta)^{6-12-3}$$

$$= (\cos\theta + i\sin\theta)^{-9}$$

$$= \cos 9\theta - i\sin 9\theta \qquad \text{when } \theta = \frac{\pi}{9}$$

$$= \cos 9\left(\frac{\pi}{9}\right) - i\sin 9\left(\frac{\pi}{9}\right)$$

$$= \cos\pi - i\sin\pi$$

$$= -1 - i(0) = -1$$

3. Prove that $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$ ·

***Solution:***

$$\text{LHS}: \left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \left[\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta} \times \frac{i}{i}\right]^4$$

$$= (i)^4 \left[\frac{\cos\theta + i\sin\theta}{i\sin\theta + i^2\cos\theta}\right]^4$$

$$= 1\left[\frac{\cos\theta + i\sin\theta}{-\cos\theta + i\sin\theta}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{-(\cos\theta - i\sin\theta)}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta)^{-1}}\right]^4$$

$$= \left[\left(\cos\theta + i\sin\theta\right)^{1+1}\right]^4$$

$$= \left[(\cos\theta + i\sin\theta)^2\right]^4$$

$$= (\cos\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta = \text{RHS}$$

4. Prove that $\left[\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n = \cos n\theta + i\sin n\theta$.

***Solution:***

Let $z = \cos\theta + i\sin\theta$

$$\Rightarrow \frac{1}{z} = \cos\theta - i\sin\theta$$

$$\text{LHS}: \left[\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n$$

$$= \left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$= \left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$= \left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$= z^n$$

$$= (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta = \text{RHS}$$

5. Show that $\left[\frac{1+\sin A + i\cos A}{1+\sin A - i\cos A}\right]^n = \cos n\left(\frac{\pi}{2} - A\right) + i\sin n\left(\frac{\pi}{2} - A\right)$.

***Solution:***

Let $z = \sin A + i\cos A$

$$\Rightarrow z = \cos\left(\frac{\pi}{2} - A\right) + i\sin\left(\frac{\pi}{2} - A\right)$$

$$\therefore \frac{1}{z} = \cos\left(\frac{\pi}{2} - A\right) - i\sin\left(\frac{\pi}{2} - A\right) = \sin A - i\cos A$$

$$\text{LHS}\left[\frac{1+\sin A+i\cos A}{1+\sin A-i\cos A}\right]^n$$

$$=\left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$=\left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$=\left[\frac{z(1+z)}{(1+z)}\right]^n$$

$$=(z)^n$$

$$=\left[\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)\right]^n$$

$$=\cos n\left(\frac{\pi}{2}-A\right)+i\sin n\left(\frac{\pi}{2}-A\right)$$

6. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ prove that

(i) $\cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$

(ii) $\sin(\theta-\phi)=\frac{1}{2i}\left[\frac{a}{b}-\frac{b}{a}\right]$

***Solution:***

Given $a = \cos\theta + i\sin\theta$

& $b = \cos\phi + i\sin\phi$

Now, $ab = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$\Rightarrow ab = \cos(\theta+\phi) + i\sin(\theta+\phi)$ .....(1)

also, $\frac{1}{ab}=(ab)^{-1}=[\cos(\theta+\phi)+i\sin(\theta+\phi)]$

$\Rightarrow \frac{1}{ab}=\cos(\theta+\phi)-i\sin(\theta+\phi)$.........(2)

$\therefore$ (1) + (2) $\Rightarrow$

$$ab+\frac{1}{ab}=\cos(\theta+\phi)+i\sin(\theta+\phi)+\cos(\theta+\phi)-i\sin(\theta+\phi)$$

$\Rightarrow ab+\frac{1}{ab}=2\cos(\theta+\phi)$

$\Rightarrow \cos(\theta+\phi)=\frac{1}{2}\left[ab+\frac{1}{ab}\right]$

(ii) $\frac{a}{b}=a(b)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)^{-1}$

$=(\cos\theta+i\sin\theta)(\cos\phi-i\sin\phi)$

$$\Rightarrow \frac{a}{b} = \cos(\theta-\phi) + i\sin(\theta-\phi) .......(3)$$

$$\text{also, } \frac{b}{a} = \left(\frac{a}{b}\right)^{-1} = \left[\cos(\theta-\phi) + i\sin(\theta-\phi)\right]^{-1}$$

$$\Rightarrow \frac{b}{a} = \cos(\theta-\phi) - i\sin(\theta-\phi) \quad ........(4)$$

$$(3) - (4) \Rightarrow$$

$$\frac{a}{b} - \frac{b}{a} = \cos(\theta-\phi) + i\sin(\theta-\phi) - \cos(\theta-\phi) + i\sin(\theta-\phi)$$

$$\Rightarrow \frac{a}{b} - \frac{b}{a} = 2i\sin(\theta-\phi)$$

$$\Rightarrow \sin(\theta-\phi) = \frac{1}{2i}\left[\frac{a}{b} - \frac{b}{a}\right]$$

7. If a = cos x + i sin x, b = cos y + i sin y prove that

(i) $\sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$

(ii) $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$

***Solution:***

Given: a = cos x + i sin x

& b = cos y + i sin y

(i) Now, ab = (cos x + i sin x) (cos y + i sin y)

$\Rightarrow$ ab = cos (x + y) + i sin (x + y)

$$\therefore \sqrt{ab} = (ab)^{1/2} = \left[\cos(x+y) + i\sin(x+y)\right]^{1/2}$$

$$\Rightarrow \sqrt{ab} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) \quad .......(1)$$

$$\text{also, } \frac{1}{\sqrt{ab}} = \left(\sqrt{ab}\right)^{-1} = \left[\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)\right]^{-1}$$

$$\Rightarrow \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right) \quad ......(2)$$

$$\therefore (1) + (2) \Rightarrow$$

$$\sqrt{ab} + \frac{1}{\sqrt{ab}} = \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right) + \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right)$$

$$\Rightarrow \sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$$

$$\text{(ii) } \frac{a}{b} = a(b)^{-1} = (\cos x + i\sin x)(\cos y + i\sin y)^{-1}$$

$$= (\cos x + i\sin x)(\cos y - i\sin y)$$

$$\Rightarrow \quad \frac{a}{b} = \cos(x-y) + i\sin(x-y)$$

$$\therefore \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/2} = \left[\cos(x-y) + i\sin(x-y)\right]^{1/2}$$

$$\Rightarrow \quad \sqrt{\frac{a}{b}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) \quad ........(3)$$

also, $$\sqrt{\frac{b}{a}} = \left(\sqrt{\frac{a}{b}}\right)^{-1} = \left[\cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right)\right]^{-1}$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right) \quad ......(4)$$

$\therefore (3) + (4) \Rightarrow$

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \cos\left(\frac{x-y}{2}\right) + i\sin\left(\frac{x-y}{2}\right) + \cos\left(\frac{x-y}{2}\right) - i\sin\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$$

8. If a = cos α + i sin α, b = cos β + i sin β and c = cos γ + i sin γ find the value of $\frac{ab}{c} - \frac{c}{ab}$.

***Solution:***

Given:

a = cos α + i sin α

b = cos β + i sin β

& c = cos γ + i sin γ

Now,

$$\frac{ab}{c} = ab(c)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)^{-1}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma - i\sin\gamma)$$

$$\Rightarrow \quad \frac{ab}{c} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) \quad ......(1)$$

also, $$\frac{c}{ab} = \left[\frac{ab}{c}\right]^{-1}$$

$$= \left[\cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)\right]^{-1}$$

$$\Rightarrow \quad \frac{c}{ab} = \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma) \quad ........(2)$$

$\therefore (1) - (2) \Rightarrow$

$$\frac{ab}{c} - \frac{c}{ab} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) - \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)$$

$$\Rightarrow \frac{ab}{c} - \frac{c}{ab} = 2i\sin(\alpha+\beta-\gamma)$$

9. If $x + \frac{1}{x} = 2\cos\theta$ prove that (i) $x^n + \frac{1}{x^n} = 2\cos n\theta$ (ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

***Solution:***

Given: $x+\frac{1}{x}=2\cos\theta$

$\frac{x^2+1}{x}=2\cos\theta$

$x^2+1=2x\cos\theta$

$x^2-2x\cos\theta+1=0$

Here a = 1, b = –2 cos θ & c = 1

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2\cos\theta\pm\sqrt{(-2\cos\theta)^2-4\times1\times1}}{2\times1}$$

$$=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(\cos^2\theta-1)}}{2}$$

$$=\frac{2\cos\theta\pm\sqrt{4(-\sin^2\theta)}}{2}$$

$$=\frac{2\cos\theta\pm i2\sin\theta}{2}$$

$$=\frac{2[\cos\theta\pm i\sin\theta]}{2}$$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

Consider $x = \cos\theta + i\sin\theta$

$\therefore x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

& $\frac{1}{x^n}=\cos n\theta - i\sin n\theta$

(i) $x^n+\frac{1}{x^n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$\Rightarrow x^n+\frac{1}{x^n}=2\cos n\theta$

(ii) $x^n-\frac{1}{x^n}=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$

$\Rightarrow x^n-\frac{1}{x^n}=2i\sin n\theta$

10. Show that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$.

***Solution:***

Let $1 + i = r(\cos\theta + i\sin\theta) = r\cos\theta + i\sin\theta$

Equating real & imaginary parts on both sides

$r\cos\theta = 1 \quad \& \quad r\sin\theta = 1$

Now, $(r\cos\theta)^2 + (r\sin\theta)^2 = (1)^2 + (1)^2$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 2$

$\Rightarrow r^2(1) = 2$

$\Rightarrow r^2 = 2 \quad \Rightarrow \boxed{r = \sqrt{2}}$

Also, $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{1}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \tan^{-1}(1)$

$\Rightarrow \boxed{\theta = \dfrac{\pi}{4}}$

$\therefore 1 + i = \sqrt{2}\left[\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right]$ .........(1)

Similarly we can prove that

$1 - i = \sqrt{2}\left[\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right]$.......(2)

LHS: $(1 + i)^n + (1 - i)^n$

$= \left[\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)\right]^n$

$= \left(\sqrt{2}\right)^n\left[\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right]^n + \left(\sqrt{2}\right)^n\left[\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right]^n$

$= \left(\sqrt{2}\right)^n\left[\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4} + \cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right]$

$= 2^{\frac{n}{2}}2\cos\dfrac{n\pi}{4}$

$= 2^{\frac{n}{2}+1}\cos\dfrac{n\pi}{4}$

$= 2^{\frac{n+2}{2}}\cos\dfrac{n\pi}{4} = \text{RHS}$

# ROOTS OF COMPLEX NUMBERS

**Definition:**

A number $\omega$ is called the $n^{th}$ root of a complex number z, if $\omega^n = z$ and we write $\omega = z^{\frac{1}{n}}$.

**Working rule to find the $n^{th}$ roots of a complex numbers:**

Step (I) : Write the given complex number in Polar form.

Step (II) : Add "2kπ" to the argument.

Step (III) : Apply Demoivre's theorem

Step (IV) : Put k = 0, 1, ....... upto (n – 1).

**Illustration:**

Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$ where $k \in I$

$$\therefore z^{\frac{1}{n}} = \{r[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)]\}^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}[\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right] \text{ where } K = 0, 1, 2, \ldots, n-1.$$

Only these values of k will give 'n' different values of $z^{\frac{1}{n}}$ provided $z \neq 0$.

## To find the $n^{th}$ roots of unity

$1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$

$$\therefore n^{th} \text{ roots of unity} = 1^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

$$= \cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right) \text{ where } k = 0, 1, 2, \ldots, n-1$$

The roots are,

for $k = 0$; $R_1 = \cos 0 + i\sin 0 = 1 + i0 = 1 = e^{i0}$

$$k=1;\ R_2 = \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n} = e^{i\frac{2\pi}{n}} = \omega \text{ (say)}$$

$$k=2;\ R_3 = \cos\frac{4\pi}{n}+i\sin\frac{4\pi}{n} = e^{i\frac{4\pi}{n}} = \left[e^{i\frac{2\pi}{n}}\right]^2 = \omega^2$$

$$k=n-1;\ R_n = \cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)\pi}{n} = e^{i\frac{2(n-1)\pi}{n}} = \omega^{n-1}$$

$\therefore$ The $n^{th}$ roots of unity are

$$e^{i0}, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots, e^{i\frac{2(n-1)\pi}{n}}$$

i.e., $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

***Result:***

If $\omega$ is $n^{th}$ roots of unity then

(i) $\omega^n = 1$

(ii) Sum of the roots is zero.

i.e $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

(iii) The roots are in G.P with common ratio ω.

(iv) The arguments are in A.P with common difference $\frac{2\pi}{n}$.

(v) The product of the roots is $(-1)^{n+1}$.

**To find cube roots of unity**

Let $x = (1)^{1/3}$

$$= (\cos 0 + i\sin 0)^{1/3}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) \text{ where } k = 0, 1, 2$$

∴ The roots are

for k = 0; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k = 1;\quad R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$k = 2;\quad R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The cube roots of unity are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

***Result:***

If we denote the second root $R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by ω then the other root ,

$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ becomes $\omega^2$

$$\text{Thus, } R_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

$$R_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

∴ The cube roots of unit are 1, ω, $\omega^2$

***Note:***

If ω is cube roots of unity then (i) $\omega^3 = 1$, (ii) $1 + \omega + \omega^2 = 0$

**Fourth roots of unity**

Let $x = (1)^{1/4}$

$$= (\cos 0 + i\sin 0)^{1/4}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/4}$$

$$= \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right) \text{ where } k = 0, 1, 2, 3$$

∴ The roots are,

for k = 0; $R_1 = \cos 0 + i \sin 0 = 1 + i0 = 1$

$$k=1;\quad R_2=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i(1)=i=\omega \text{ (say)}$$

$$k=2;\quad R_3=\cos\pi+i\sin\pi=-1+i(0)=-1=\omega^2$$

$$k=3;\quad R_4=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=0+i(-1)=-i=\omega$$

The fourth roots of unity are 1, i, – 1, – i (i.e.) 1, $\omega$, $\omega^2$, $\omega^3$.

***Note:***

(i) The sum of the fourth roots of unity is zero. i.e $1+\omega+\omega^2+\omega^3=0$ and $\omega^4=1$.

(ii) The value of $\omega$ used in cube roots of untiy and in fourth roots of unity are different.

## Sixth roots of unity

Let $x=(1)^{1/6}$

$$=(\cos 0+i\sin 0)^{1/6}$$

$$=(\cos 2k\pi+i\sin 2k\pi)^{1/6}$$

$$=\cos\left(\frac{2k\pi}{6}\right)+i\sin\left(\frac{2k\pi}{6}\right) \text{ where } k=0,1,2,3,4,5$$

$\therefore$ The six roots are

for k = 0; $R_1=\cos 0+i\sin 0=e^{i0}=1$

$$k=1;\quad R_2=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}=e^{i\frac{2\pi}{6}}=\omega$$

$$k=2;\quad R_3=\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}=e^{i\frac{4\pi}{6}}=\omega^2$$

$$k=3;\quad R_4=\cos\frac{6\pi}{6}+i\sin\frac{6\pi}{6}=e^{i\frac{6\pi}{6}}=\omega^3$$

$$k=4;\quad R_5=\cos\frac{8\pi}{6}+i\sin\frac{8\pi}{6}=e^{i\frac{8\pi}{6}}=\omega^4$$

$$k=5;\quad R_6=\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}=e^{i\frac{10\pi}{6}}=\omega^5$$

$\therefore$ The sixth roots of unity are $e^{i0}, e^{i\frac{2\pi}{6}}, e^{i\frac{4\pi}{6}}, e^{i\frac{6\pi}{6}}, e^{i\frac{8\pi}{6}}, e^{i\frac{10\pi}{6}}$

i.e, 1, $\omega$, $\omega^2$, $\omega^3$, $\omega^4$, $\omega^5$.

***Note:***

The sum of the sixth roots of unity is zero. i.e $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5=0$ and $\omega^6=1$

Note: $1=\cos 0+i\sin 0$

$-1=\cos\pi+i\sin\pi$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$

$$-i=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$$

### WORKED EXAMPLES

### PART – A

1. If $\omega$ is a cube roots of unity, find the value of $\omega^4+\omega^5+\omega^6$.

***Solution:***

If ω is cube roots of unity then $\omega^3 = 1$.

$$\therefore\ \omega^4 + \omega^5 + \omega^6 = \omega^3 . \omega + \omega^3 . \omega^2 + \omega^3 . \omega^3$$
$$= (1)\,\omega + (1)\,\omega^2 + (1)(1)$$
$$= \omega + \omega^2 + 1$$
$$= 1 + \omega + \omega^2 = 0$$

2. Simplify: $(1 + \omega)(1 + \omega^2)$ where ω is cube roots of unity.

***Solution:***

$$(1 + \omega)(1 + \omega^2) = 1 + \omega^2 + \omega + \omega^3$$
$$= [1 + \omega + \omega^2] + \omega^3$$
$$= 0 + 1$$
$$= 1$$

3. Solve: $x^2 - 1 = 0$

***Solution:***

Given: $x^2 - 1 = 0$

$$x^2 = 1$$
$$x = (1)^{1/2} = [\cos 0 + i \sin 0]^{1/2}$$
$$= [\cos 2k\pi + i \sin 2k\pi]^{1/2}$$
$$= \cos\left(\frac{2k\pi}{2}\right) + i \sin\left(\frac{2k\pi}{2}\right) \text{ where } k = 0,1$$

4. Find the value of $(i)^{1/3}$.

***Solution:***

Let $x = (i)^{1/3}$

$$= \left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right]^{1/3}$$
$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i \sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/3}$$
$$= \cos\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) + i \sin\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) \quad \text{where } k = 0, 1, 2.$$

5. Find the value of $(-1)^{1/3}$.

***Solution:***

Let $x = (-1)^{1/3}$

$$= (\cos\pi + i \sin\pi)^{1/3}$$
$$= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{1/3}$$
$$= \cos\frac{1}{3}(2k\pi + \pi) + i \sin\frac{1}{3}(2k\pi + \pi) \text{ where } k = 0, 1, 2$$

6. Find the value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

***Solution:***

We have $\frac{-1+i\sqrt{3}}{2}=\frac{-1}{2}+i\frac{\sqrt{3}}{2}$ = cos 120° + i sin 120°

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^3=(\cos 120^o+i\sin 120^o)^3$$

$$=\cos 360^o+i\sin 360^o$$

$$=1+i(0)$$

$$=1$$

**PART – B**

1. Find the cube roots of unity.

***Solution:***

Let ‘x’ be the cube roots of unity.

i.e. $x^3 = 1$

$$x=(1)^{1/3}$$

$$=(\cos 0+i\sin 0)^{1/3}$$

$$=(\cos 2k\pi+i\sin 2k\pi)^{1/3}$$

$$=\cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}\ \text{ where } k=0,1,2$$

For k = 0; x = cos 0 + i sin 0 = 1 + i0 = 1

$$k=1;\ x=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$k=2;\ x=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

2. Find all the values of $(i)^{2/3}$.

***Solution:***

Let $x=(i)^{2/3}$

$$=\left[\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right]^{2/3}$$

$$=\left[\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^2\right]^{1/3}$$

$$=\left[\cos 2\left(\frac{\pi}{2}\right)+i\sin 2\left(\frac{\pi}{2}\right)\right]^{1/3}$$

$$=[\cos\pi+i\sin\pi]^{1/3}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$$

$$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right) \quad \text{where } k = 0, 1, 2$$

when $k = 0$ ; $x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$k = 1$; $x = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$

$k = 2$; $x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

3. Solve: $x^2 + 16 = 0$

***Solution:***

Given: $x^2 + 16 = 0$

$x^2 = -16 = 16 \times -1$

$x = (16)^{1/2}(-1)^{1/2}$

$= 4[\cos\pi + i\sin\pi]^{1/2}$

$= 4[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/2}$

$$= 4\left[\cos\left(\frac{2k\pi + \pi}{2}\right) + i\sin\left(\frac{2k\pi + \pi}{2}\right)\right] \text{ where } k = 0, 1$$

when $k = 0$; $x = 4\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$k = 1$; $x = 4\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$

4. If $\omega$ is the cube roots of unity then prove that $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$.

***Solution:***

If $\omega$ is the cube roots of unity then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

LHS : $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

$= (1 + \omega^2 - \omega)^5 + (1 + \omega - \omega^2)^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$

$= (-2\omega)^5 + (-2\omega^2)^5$

$= (-2)^5 \omega^5 + (-2)^5 (\omega^2)^5$

$= -32\omega^2 - 32\omega$

$= -32(\omega^2 + \omega)$

$= -32(-1) = 32 = \text{RHS}$

## PART – C

1. Solve: $x^7 + 1 = 0$

***Solution:***

Given: $x^7 + 1 = 0$

$$x^7 = -1$$

$$x = (-1)^{1/7}$$

$$= (\cos\pi + i\sin\pi)^{1/7}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/7}$$

$$= \cos\left(\frac{2k\pi + \pi}{7}\right) + i\sin\left(\frac{2k\pi + \pi}{7}\right) \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6,$$

$\therefore$ The values are

when $k = 0;\; x = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$k = 1;\; x = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$

$k = 2;\; x = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

$k = 3;\; x = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7}$

$k = 4;\; x = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$k = 5;\; x = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$

$k = 6;\; x = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$

2. Solve: $x^6 - 1 = 0$

***Solution:***

Given: $x^6 - 1 = 0$

$$x^6 = 1$$

$$x = (1)^{1/6}$$

$$= (\cos 0 + i\sin 0)^{1/6}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/6}$$

$$= \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6} \quad \text{where } k = 0, 1, 2, 3, 4, 5$$

$\therefore$ The values are,

when $k = 0$; $x = \cos 0 + i\sin 0$

$k = 1$; $x = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}$

$k = 2$; $x = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}$

$k = 3$; $x = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}$

$k = 4$; $x = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6}$

$k = 5$; $x = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}$

3. Solve: $x^8 + x^5 + x^3 + 1 = 0$

***Solution:***

Given: $x^8 + x^5 + x^3 + 1 = 0$

$x^5 (x^3 + 1) + 1 (x^3 + 1) = 0$

$(x^5 + 1) (x^3 + 1) = 0$

$x^5 + 1 = 0$ ; $x^3 + 1 = 0$

Case (i)

$x^5 + 1 = 0$

$x = (-1)^{1/5}$

$= (\cos\pi + i\sin\pi)^{1/5}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/5}$

$= \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right)$ where $k = 0, 1, 2, 3, 4$

The roots are,

when $k = 0$; $x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$k = 1$; $x = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$k = 2$; $x = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$

$k = 3$; $x = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$k = 4$; $x = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

Case (ii) :

$x^3 + 1 = 0$

$x = (-1)^{1/3}$

$= (\cos\pi + i\sin\pi)^{1/3}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/3}$

$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right)$ where k = 0, 1, 2

when $k = 0;\ x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$k = 1;\ x = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$

$k = 2;\ x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

4. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and also prove that the product of the four values is 1.

***Solution:***

Let $a + ib = \frac{1}{2} + i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$ ........(1)

Here $a = \frac{1}{2}$ & $b = \frac{\sqrt{3}}{2}$

Modulus:

$$r = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

Arugument:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left[\frac{\sqrt{3}/2}{1/2}\right] = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$\therefore$(1) becomes,

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\Rightarrow \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$$

$$= \left[\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}$$

$$= \left[\cos 3\left(\frac{\pi}{3}\right) + i\sin 3\left(\frac{\pi}{3}\right)\right]^{1/4}$$

$$= [\cos\pi + i\sin\pi]^{1/4}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$$

$$= \cos\left(\frac{2k\pi + \pi}{4}\right) + i\sin\left(\frac{2k\pi + \pi}{4}\right) \quad \text{where } k = 0, 1, 2, 3$$

∴ The values are,

when $k = 0$; $R_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$k = 1$; $R_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$k = 2$; $R_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$k = 3$; $R_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

**Product of the four values**

$R_1 \times R_2 \times R_3 \times R$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= \cos\frac{16\pi}{4} + i\sin\frac{16\pi}{4}$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1 + i(0) = 1$$

## EXERCISE

## PART – A

1. If $z_1 = -1 + 2i$ and $z_2 = -3 + 4i$ find $3z, - 4z_2$.
2. If $z_1 = (2, 3)$ and $z = (5, 7)$ find $4z_1 + 3z_2$.
3. If $z_1 = (-3, 5)$ and $z_2 = (1, -2)$ find $z_1 z_2$.
4. If $z_1 = 1 + i$ and $z_2 = 1 - i$ find $z_1 / z_2$.
5. If $z_1 = 2 + i$ and $z_2 = 1 + i$ find $z_2 / z_1$.
6. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{4+3i}$ (ii) $\frac{2}{3-i}$ (iii) $(4+5i)(5+7i)$

7. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1}{2-i}$ (ii) $\frac{1}{2+3i}$ (iii) $\frac{1}{i-3}$ (iv) $\frac{1+i}{1-i}$

8. Find the complex conjugate of the following:

   (i) $(2-3i)(7+11i)$ (ii) $\frac{4}{1-i}$ (iii) $\frac{1-i}{1+i}$ (iv) $\frac{2}{i-5}$

9. Find the modulus and argument (or) amplitude of the following:

   (i) $\sqrt{3}+i$ (ii) $-1+i$ (iii) $\sqrt{3}-i$ (iv) $1-\sqrt{-3}$ (v) $\frac{1}{2}+i\frac{\sqrt{3}}{2}$

   (vi) $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$ (vii) $1-i\sqrt{3}$

10. Find the distance between the following two complex numbers

    (i) $2 + 3i$ and $3 - 2i$ (ii) $4 + 3i$ and $5 - 6i$ (iii) $2 - 3i$ and $5 + 7i$ (iv) $1 + i$ and $3 - 2i$

11. State DeMoivre's theorem.
12. Simplify the following:

    (i) $(\cos\theta + i\sin\theta)^3 (\cos\theta + i\sin\theta)^{-4}$

    (ii) $(\cos\phi + i\sin\phi)^5 (\cos\phi + i\sin\phi)^{-6}$

    (iii) $(\cos\theta - i\sin\theta)^4 (\cos\theta + i\sin\theta)^7$

13. Find the value of the following:

    (i) $\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta - i\sin 3\theta}$ (ii) $\frac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

    (iii) $\frac{\cos 10\theta + i\sin 10\theta}{\cos 7\theta + i\sin 7\theta}$ (iv) $\frac{\cos 4\theta + i\sin 4\theta}{\cos\theta + i\sin\theta}$

    (v) $\frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^4}$ (vi) $\frac{\cos 6\theta + i\sin 6\theta}{(\cos\theta - i\sin\theta)^4}$

14. Simplify: $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$

15. Simplify: $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

16. Find the product of $5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ and $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

17. Find the product of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

17.(a) If $z_1 = 5\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ find $Z_1Z_2$.

18. If $x = \cos\theta + i\sin\theta$ find $x-\frac{1}{x}$.

19. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$ find ab.

20. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ find $\frac{b}{a}$.

21. If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$ find $\sqrt{ab}$.

22. If ω is the cube root of unity find the value of $1 + \omega^2 + \omega^4$.

23. If ω is the fourth root of unity find the value $\omega^4 + \omega^5 + \omega^6 + \omega^7$.

24. If ω is the six root of unity find the value of $\omega^2 + \omega^4 + \omega^6$.

25. Solve: $x^2 + 1 = 0$

26. Find the value of (i) $(i)^{1/2}$ (ii) $(1)^{1/3}$ (iii) $(-1)^{1/2}$

27. Find the value of $\left(\frac{-1-i\sqrt{3}}{2}\right)^3$.

**PART – B**

1. Express the following complex numbers in a + ib form.

   (i) $\frac{1}{1+i}+\frac{1}{1-i}$ (ii) $\frac{2+3i}{4-i}$ (iii) $\frac{1+i}{(1-i)^2}$ (iv) $\frac{4+3i}{1-i}$

2. Find the real and imaginary parts of the following complex numbers

   (i) $\frac{1+2i}{1-i}$ (ii) $\frac{(2-i)^2}{1+i}$ (iii) $\frac{2+i}{1+4i}$

3. Find the conjugate the following complex numbers

   (i) $\frac{13}{11+12i}$ (ii) $\frac{1+i}{1-i}$ (iii) $\frac{1}{2-i}+\frac{1}{2+i}$

4. Find the modulus and amplitude of the following complex numbers

   (i) $\frac{1+i}{1-i}$ (ii) $\frac{\sqrt{3}}{2}+i\frac{\sqrt{3}}{2}$ (iii) $2+2\sqrt{3}i$ (iv) $-\sqrt{2}+\sqrt{2}i$

5. Show that the following complex numbers are collinear.

   (i) 1 + 3i, 5 + i, 3 + 2i

   (ii) 4 + 2i, 7 + 5i, 9 + 7i

   (iii) 1 + 3i, 2 + 7i, – 2 –9i

6. Show that the following complex numbers form a parallelogram.

(i) 2 – 2i, 8 + 4i, 5 + 7i, – 1 + i

(ii) 3 + i, 2 + 2i, – 2 + i, – 1

(iii) 1 – 2i, –1 + 4i, 5 + 8i, 7 + 2i

7. If x = cos θ + i sin θ find the value of

(i) $x^n + \frac{1}{x^n}$ (ii) $x^m - \frac{1}{x^m}$ (iii) $x^3 + \frac{1}{x^3}$ (iv) $x^5 + \frac{1}{x^5}$ (v) $x^2 - \frac{1}{x^2}$ (vi) $x^8 - \frac{1}{x^8}$

8. If a = cos x + i sin x, b = cos y + i sin y find $ab - \frac{1}{ab}$.

9. If x = cos 2α + i sin 2α, y = cos 2β + i sin 2β find $\sqrt{xy}$.

10. If x = cos α + i sin α, y = cos β + i sin β find $x^m y^n$.

11. Find the cube roots of 8.

12. Find the all the values of $(-1)^{2/3}$.

## PART – C

1. Find the real and imaginary parts of the following complex numbers

(i) $\frac{(1+i)(1+2i)}{1+3i}$ (ii) $\frac{(1+2i)^3}{(1+i)(2-i)}$ (iii) $\left(\frac{1-i}{1+i}\right)^3$ (iv) $\frac{3}{4+3i} + \frac{i}{3-4i}$

(vii) $\frac{3}{3+4i} + \frac{i}{5-2i}$ (viii) $\frac{4}{3+2i} + \frac{2}{5-4i}$

(ix) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$ (x) $\frac{1}{1+\cos\theta + i\sin\theta}$

2. Express the following complex numbers in a + ib form.

(i) $\frac{(1+i)(1-2i)}{(1+3i)}$ (ii) $\frac{(1+i)(1+2i)}{(1+4i)}$ (iii) $\frac{(1+i)(3+i)^2}{(2-i)^2}$

(iv) $\frac{3}{4+3i} + \frac{i}{3-4i}$ (v) $\frac{2+3i}{1-i}$ (vi) $\frac{7-5i}{(2+3i)^2}$

3. Find the conjugate of the following complex numbers

(i) $\frac{(1+i)(2-i)}{(2+i)^2}$ (ii) $\frac{(1+i)(2+i)}{(3+i)}$ (iii) $\frac{1-i}{3+2i}$ (iv) $\frac{3+i}{2+5i}$ (v) $\frac{5-i}{2-3i}$

4. Find the modulus and argument of the following complex numbers

(i) $\frac{1+\sqrt{3}i}{1+i}$ (ii) $\frac{2-i}{3+7i}$ (iii) $\frac{1+i\sqrt{3}}{1-i}$

(iv) $\frac{(1+i)(1+2i)}{1+3i}$ (v) $\frac{i-3}{i-1}$ (vi) $\frac{1-i}{1+i}$

5. Prove that the following complex numbers form a square.

(i) 9 + i, 4 + 13i, – 8 + 8i, – 3 – 4i (ii) 2 + i, 4 + 3i, 2 + 5i, 3i

(iii) – 1, 3i, 3 + 2i, 2 – i (iv) 4 + 5i, 1 + 2i, 4 – i, 7 + 2i

6. Show that the following complex numbers form a rectangle.

(i) $1 + 2i, -2 + 5i, 7i, 3 + 4i$ (ii) $4 + 3i, 12 + 9i, 15 + 5i, 7 - i$

(iii) $1 + i, 3 + 5i, 4 + 4i, 2i$ (iv) $8 + 4i, 5 + 7i, -1 + i, 2 - 2i$

7. Show that the following complex number form a rhombus.

(i) $8 + 5i, 16 + 11i, 10 + 3i, 2 - 3i$ (ii) $6 + 4i, 4 + 5i, 6 + 3i, 8 + i$

(iii) $1 + i, 2 + i, 2 + 2i, 1 + 2i$

8. Simplify the following using De Moivre's theorem

(i) $\dfrac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(ii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^3(\cos 3\theta + i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^{-3}}$

(iii) $\dfrac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^2(\cos 5\theta - i\sin 5\theta)^{-6}}$

(iv) $\dfrac{(\cos 3\theta + i\sin 3\theta)^2(\cos 4\theta - i\sin 4\theta)^3}{(\cos\theta + i\sin\theta)^3}$ when $\theta = \dfrac{\pi}{9}$

(v) $\dfrac{(\cos x - i\sin x)^3(\cos 3x + i\sin 3x)^5}{(\cos 2x - i\sin 2x)^5(\cos 5x + i\sin 5x)^7}$ when $x = \dfrac{2\pi}{13}$

(vi) $\dfrac{(\cos 5\theta - i\sin 5\theta)\,(\cos 2\theta - i\sin 2\theta)^{-3}}{(\cos\theta + i\sin\theta)^5(\cos 3\theta + i\sin 3\theta)^{-5}}$ when $\theta = \dfrac{2\pi}{11}$

9. Show that $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^4 = 1$

10. Show that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^3 = \cos 3\theta + i\sin 3\theta$

11. Prove that $\left[\dfrac{1+\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}\right]^8 = 1$.

12. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$ find the value of $\dfrac{ab}{c} + \dfrac{c}{ab}$.

13. If $x = \cos 3\alpha + i\sin 3\alpha$, $y = \cos 3\beta + i\sin 3\beta$ prove that

(i) $\sqrt[3]{xy} + \dfrac{1}{\sqrt[3]{xy}} = 2\cos(\alpha + \beta)$ (ii) $\sqrt[3]{xy} - \dfrac{1}{\sqrt[3]{xy}} = 2i\sin(\alpha + \beta)$

14. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and if $x + y + z = 0$ show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

15. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ and $z = \cos\gamma + i\sin\gamma$ and if $x + y + z = 0$ prove that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$

16. If $x + \dfrac{1}{x} = 2\cos\theta$ and $y + \dfrac{1}{y} = 2\cos\phi$ show that $\dfrac{x^m}{y^m} + \dfrac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

17. If x = cos α + i sin α, y = cos β + i sin β prove that $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$.

18. If 'n' is a positive integer, prove that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\dfrac{n\pi}{3}$.

19. If 'n' is a positive integer prove that $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 2^{n+1}\cos\dfrac{n\pi}{6}$.

20. Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{n\theta}{2}\right)$.

21. Solve: $x^4 + 1 = 0$

22. Solve: $x^5 + 1 = 0$

23. Solve: $x^6 + 1 = 0$

24. Solve: $x^4 - 1 = 0$

25. Solve: $x^5 - 1 = 0$

26. Solve: $x^7 - 1 = 0$

27. Solve: $x^5 + x^3 + x^2 + 1 = 0$

28. Solve: $x^8 - x^5 + x^3 - 1 = 0$

29. Solve: $x^7 + x^4 + x^3 + 1 = 0$

30. Solve: $x^7 - x^4 + x^3 - 1 = 0$

## ANSWERS

## PART – A

(1) 9 – 10i (2) (23, 33) (3) 7 – i (4) i (5) $\dfrac{3+i}{5}$ (6) (i) $\dfrac{4}{25} - \dfrac{3i}{25}$ (ii) $\dfrac{3}{5} + \dfrac{i}{5}$ (iii) $-15 + 53i$

(7) (i) $Re(z) = \dfrac{1}{5}, Im(z) = \dfrac{1}{5}$ (ii) $Re(z) = \dfrac{2}{13}, Im(z) = \dfrac{-3}{13}$

(iii) $Re(z) = \dfrac{-3}{10}, Im(z) = \dfrac{-1}{10}$ (iv) $Re(Z) = 0, Im(z) = 1$

(8) (i) $\overline{Z} = 47 - i$ (ii) $\overline{Z} = 2(1-i)$ (iii) $\overline{Z} = i$ (iv) $\overline{Z} = \dfrac{-5+i}{13}$

(9)

(i) $r = 2; \theta = \dfrac{\pi}{6}$ (ii) $r = \sqrt{2}; \theta = -\dfrac{\pi}{4}$ (iii) $r = 2; \theta = -\dfrac{\pi}{6}$

(iv) $r = 2; \theta = -\dfrac{\pi}{3}$ (v) $r = 1; \theta = \dfrac{\pi}{3}$ (vi) $r = 1; \theta = \dfrac{\pi}{3}$ (vii) $r = 2; \theta = -\dfrac{\pi}{3}$

10) (i) $\sqrt{26}$ (ii) $\sqrt{82}$ (iii) $\sqrt{109}$ (iv) $\sqrt{13}$

12) (i) cos θ – i sin θ (ii) cos 11ϕ + i sin 11ϕ (iii) cos 3θ + i sin 3θ

13) (i) cos 8θ + i sin 8θ (ii) cos 5θ + i sin 5θ (iii) cos 3θ + i sin 3θ

(iv) cos 3θ + i sin 3θ (v) cos 4θ + i sin 4θ (vi) cos 10θ + i sin 10θ

14) – 1 15) –1 16) $5(-1-i\sqrt{3})$ 17) i 17(a) – 15 18) 2i sin θ

19) $\cos(\theta + \phi) + i\sin(\theta + \phi)$ 20) $\cos(\beta - \alpha) + i\sin(\beta - \alpha)$

21) $\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)$ 22) 0 23) 0 24) $-\omega - \omega^3 - \omega^5$

25) $x = \cos\left(\frac{2k\pi + \pi}{2}\right) + i\sin\left(\frac{2k\pi + \pi}{2}\right)$ where k = 0, 1

26) (i) $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

(ii) $\cos 0 + i\sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(iii) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2},\ \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

27) 1

## PART – B

1) (i) 1 (ii) $\frac{11+10i}{17}$ (iii) $\frac{-1+i}{2}$ (iv) $\frac{1+7i}{2}$

2) (i) $Re(Z) = -\frac{1}{2},\quad I_m(Z) = \frac{3}{2}$

(ii) $Re(Z) = \frac{1}{2},\quad I_m(Z) = \frac{-9}{2}$

(iii) $Re(Z) = \frac{6}{17},\quad I_m(Z) = -\frac{7}{17}$

3) (i) $\overline{Z} = \frac{13}{23}(11+12i)$ (ii) $\overline{Z} = -i$ (iii) $\overline{Z} = \frac{4}{5}$

4) (i) $r = 1, \theta = \infty$ (ii) $r = \frac{3}{2}, \theta = 45^o$

(iii) $r = 4;\ \theta = \frac{\pi}{3}$ (iv) $r = 2,\ \theta = \frac{-\pi}{4}$

7) (i) $2\cos n\theta$ (ii) $2i\sin m\theta$ (iii) $2\cos 3\theta$ (iv) $2\cos 5\theta$ (v) $2i\sin 2\theta$ (vi) $2i\sin 8\theta$

8) $2i\sin(x - y)$ 9) $\cos(x + y) + i\sin(x - y)$ 10) $\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

11) $2[\cos 0 + i\sin 0],\ 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right],\ 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$

12) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3},\ \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}$

## PART – C

1) (i) $Re(z)=\frac{4}{5}, Im(z)=\frac{3}{5}$ (ii) $Re(z)=\frac{-7}{2}, Im(z)=\frac{1}{2}$

(iii) $Re(z)=0, \ Im(z)=1$ (iv) $Re(z)=\frac{8}{25}, \ Im(z)=-\frac{6}{25}$

(v) $Re(z)=1, \ Im(z)=\frac{1}{2}$ (vi) $Re(z)=\frac{26}{41}, \ Im(z)=-\frac{29}{41}$

(vii) $Re(z)=\frac{366}{725}, Im(z)=\frac{-298}{725}$ (viii) $Re(z)=\frac{622}{533}, \ Im(z)=\frac{-224}{533}$

(ix) $Re(z)=\sqrt{3}, \ Im(z)=1$ (x) $Re(z)=\frac{1}{2}, \ Im(z)=-\frac{1}{2}\tan\frac{\theta}{2}$

2) (i) $\frac{8-9i}{29}$ (ii) $\frac{193+149i}{41}$ (iii) $\frac{1}{2}-i$

(iii) $\frac{23-24i}{65}$ (v) $\frac{17+7i}{26}$ (vi) $-1$

3) (i) $\overline{Z}=\frac{13}{25}+\frac{9i}{25}$ (ii) $\overline{Z}=\frac{3}{5}-\frac{4i}{5}$ (iii) $\overline{Z}=\frac{1}{13}+\frac{5i}{13}$

(iv) $\overline{Z}=\frac{11}{29}-\frac{13}{29}i$ (v) $\overline{Z}=1-i$

4) (i) $|Z|=2, \ \theta=\tan^{-1}(1-\sqrt{3})$ (ii) $|Z|=\frac{\sqrt{290}}{58}, \ \theta=\tan^{-1}(17)$

(iii) $|Z|=\sqrt{2}, \ \theta=\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (iv) $|Z|=\sqrt{\frac{5}{2}}, \ \theta=\tan^{-1}=(9)$

(v) $|Z|=\sqrt{5}\ , \theta=\tan^{-1}\left(\frac{1}{2}\right)$ (vi) $|Z|=1, \theta=-\frac{\pi}{2}$

8) (i) $\cos 19\theta - i\sin 19\theta$ (ii) $\cos 15\theta - i\sin 15\theta$ (iii) $\cos 107\theta - i\sin 107\theta$

(iv) $-1$ (v) 1 (vi) 1

12) $2\cos(\alpha+\beta-\gamma)$

21) $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}, \ \cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}, \ \cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}, \ \cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

22) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}, \ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}, \ \cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}, \ \cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}, \cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

23) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, \ \cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6}, \ \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}, \ \cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6},$

$\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}, \ \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$

24) $\cos 0+i\sin 0, \cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}, \cos\frac{4\pi}{4}, i\sin\frac{4\pi}{4}, \ \cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}$

25) $\cos 0+i\sin 0, \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5}, i\sin\frac{4\pi}{5}, \ \cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$

26) $\cos 0+i\sin 0, \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}, \cos\frac{4\pi}{7}, i\sin\frac{4\pi}{7}, \ \cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}, \cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}, \cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

27) Case (i) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{3\pi}{3}+i\sin\frac{3\pi}{3}$, $\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

Case (ii) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$

28) Case (i) $\cos 0+i\sin 0$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

Case (ii) $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$, $\cos\frac{5\pi}{5}+i\sin\frac{5\pi}{5}$,

$\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$, $\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$

29) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$.

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$

30) $\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)$, $k=0,1,2,3$

$\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)$, $k=0,1,2$

# MATRICES

**Minor –** Minor is the determinate value which is obtained by deleting row & coloumn of the particular element and denoted by the symbol ........, i-rows j-coloum.

**Ex :** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 8 \\ 5 & 6 & 1 \end{bmatrix}$

$$M_{21} = \begin{vmatrix} 1 & 3 \\ 6 & 1 \end{vmatrix} = 1 - 18 = -17$$

$$M_{32} = \begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = 16 - 12 = 4$$

**Upper triangular Matrix –** A matrix is said to be upper triangular if the elements below the main diagoned are zeros.

**Ex.** $\begin{bmatrix} 1 & 5 & 9 \\ 0 & 3 & 7 \\ 0 & 0 & 8 \end{bmatrix}$

Elementary transformations : – The following operations three of which refer to rows are known as elementary transformations.

I. The interchange of any two rows (Rij)

II. The multiplication of any row by a non-zero scalar (kRi)

III. The addition of a constant multiple of the elements of any row to the corresponding elements of any other row (Ri + kRj)

**Equivalent matrix –** Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.

**Rank of a matrix :** A matrix is said to be of rank 'r' if

(i) It has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than 'r' varishes.

The rank of a matrix A shall be denoted by the symbol e(A).

**Working Rule :**

**Step – I :** Conver the matrix to the upper triangular form.

**Step – II :** The no.of non-zero rows is the rank of the matrix

**Example – 1 :**

Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

**Solution :**

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ -3 & 1 & 2 \end{bmatrix} \rightarrow R_2 + 2R_1$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}_{\rightarrow R_3 + R_1}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}_{\rightarrow 2R_3 - R_2}$$

$\rho(A) = 2$

**Consistency :** A system of equatiars are said to be consistent if either they will have unique solution on many solution and sid to be inconsistent if they will have no solution.

| | | |
|---|---|---|
| $2x + 3y = 8$ | $x + 2y = 5$ | $x - y = 10$ |
| $x - 2y = 4$ | $2x + 4y = 10$ | $3x - 3y = 15$ |
| (unique solution) | (many soluion) | (No solution) |

Consistency of a system of linear equations : -

Consider a system of m linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ........ + a_{1x}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ........ + a_{2x}x_n = b_2 \\ .................................................. \\ .................................................. \\ am_1x_1 + am_2x_2 + ...... + a_{mn}x_n = b_m \end{array}\right\} \quad .....(1)$$

Containing the n unknows $x_1, x_2.....x_n$.

Writing the above equations in matrix form we get.

$$A = \begin{bmatrix} a_{11} \; a_{12} \; ........ \; a_{1n} \\ a_{21} \quad a_{22} .......a_{2n} \\ ......................... \\ .......................... \\ a_{m1} \quad a_{m2} .......a_{mn} \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$C = \begin{bmatrix} A \;\vdots\; B \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} \;\; a_{12} \; ...... \; a_{1n} \; ..... \; b_1 \\ a_{21} \;\; a_{22} \; ...... \; a_{2n} \; ..... \; b_2 \\ ...................................... \\ a_{m1} \;\; a_{m2} \; ...... \; a_{mn} \; ... \; b_m \end{bmatrix}$$

A is the co-efficient matrix and

C is called agumented matrix

**Rouche's Theorem : (Without proof)**

The system of equations (1) is consistant if and only if the co-efficient matrix A and the augmented matrix C are of some rank otherwise the system is inconsistent.

Procedure to test the consistency of a system of equations in $x$ unknows.

Find the ranks of the co-efficient matrix A and the augmented matrix 'C' by reducing to the upper triangular form by elementary row operations.

(a) Consistant equations : If Rank A = Rank C

(i) Unique solution Rank A = Rank C = n

Where n = number of unknowns.

(ii) Infinite solution : Rank A = Rank C = r. r < n.

(b) Inconstant equations if Rank A ≠ Rank C

**Example – 2 :**

Show that the equations

$2x + 6y = -11$, $6x + 20y - 6z = -3$, $6y - 18z = -1$ are not consistant.

**Solution :**

Writing the above equations in matrix form

$$\underbrace{\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}}_{B}, \qquad AX = B$$

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \quad B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} \quad C = [A : B]$$

$$C = \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \rightarrow R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix} \rightarrow R_3 - 3R_2$$

The rank of C is 3

and rank of A is 2

Rank A ≠ Rank C.

∴ The system of equations are not consistant

**Example – 3 :**

Test consistency and solve :

$5x + 3y + 7z = 4$

$3x + 2by + 2z = 9$

$7x + 2y + 10z = 5$

**Solution :**

Writing the above equations in matrix form

$$\underbrace{\begin{bmatrix} 5 & 3 & 7 \\ 3 & 2b & 2 \\ 7 & 2 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}}_{B}, \qquad AX = B, C = [A : B]$$

$$C = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 2b & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 3 & 2b & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \frac{1}{5}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & \frac{-11}{5} & \frac{1}{5} & : & \frac{-3}{5} \end{bmatrix} \begin{matrix} \\ \rightarrow R_2 - 3R_1 \\ \rightarrow R_3 - 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} & : & \frac{33}{5} \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \rightarrow R_3 + \frac{1}{11} R_Z$$

Here Rank of A = Rank of C.

Hence the equations are consistent.

But the rank is less than 3 i.e. the number of unknows.

So its solutions are infinite

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{33}{5} \\ 0 \end{bmatrix}$$

$$x + \frac{3}{5} y + \frac{7}{5} z = \frac{4}{5}$$

$$\frac{121}{5} y - \frac{11}{5} z = \frac{33}{5} \text{ or } 11y - z = 3$$

Let z = k, 11y – k = 3 or y = $\frac{3}{11} + \frac{k}{11}$

$$x + \frac{3}{5}\left[\frac{3}{11} + \frac{k}{11}\right] + \frac{7}{5} k = \frac{4}{5} \text{ or } x = \frac{-16}{11} k + \frac{7}{11}$$

**Example – 4 :**

Determine the values of $\lambda$ & $\mu$ so that the following equations have (i) no solution (ii) a unique solution (iii) infinite number of solutions.

$x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$

**Solution :**

Writing the above equations in matrix form we have

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & z & \lambda \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 6 \\ 10 \\ \mu \end{pmatrix}}_{B}$$

$\therefore$ AX = B

C = [A : B]

$$C = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 3 & \lambda & : & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix} \begin{matrix} \\ \to R_Z - R_1 \\ \to R_Z - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix} \to R_3 - R_2$$

(i) There is no, solution = b $\rho(A) \neq \rho(C)$

i.e. $\lambda - 3 = 0$ or $\lambda = 3$ & $\mu - 10 \neq 0$ or $\mu \neq 10$

(ii) There is a unique solution if $\rho(A) = \rho(C) = 3$

i.e., $\lambda - 3 \neq 0$ or $\lambda \neq 3$ and $\mu$ have any value

(iii) There are infinite solution of $\rho(A) = \rho(C) = 2$

$\lambda - 3 = 10$ or $\lambda = 3$ and $\mu - 10 = 0$ or $\mu = 10$

## *Assignments*

**1.** Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

**2.** Test the consistency & solve

$4x - 5y + z = 2$

$3x + y - 2z = 9$

$x + 4y + z = 5$

**3.** Determine the values of $a$ & b for which the system of equations

$3x - 2y + z = b$

$5x - 8y + 9z = 3$

$2x + y + az = -1$

(i) has a unique solution (ii) has no solution (iii) has infinite solution.

❒ ❒ ❒

# LINEAR DIFFERENTIAL EQUATIONS

**Introduction :**

The Mathematical formulation of many problems in science, Engineering and Economics gives rise to differential Equations.

For example : The problem of motion of a satellite

- The flow of fluids.
- The flow of current in an electric circuit
- The growth of population
- The Conduction of heat in rod etc leads to differential equations

**Definition of Differential Equation :**

A differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

There are two types of Differential Equation

1. Ordinary differential Equation
2. Partial differential Equation

**Example :**

(a) $\frac{dy}{dx} + y = x^2$

(b) $\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$

(c) $\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial t}\right)^2 = 4$

**Linear differential Equation :**

Linear differential Equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The differential Equation of the form

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1} y}{dx^{n-1}} + k_2\frac{d^{n-2} y}{dx^{n-2}} + ...... + k_n y = X \qquad ........(1)$$

Is known as linear differential Equation with constant coefficients. Where $k_1, k_2 ........ k_n$ are constant, X is the function of x.

There are two types of linear differential Equation

1. Homogeneous LDE
2. Non Homogeneous LDE

**Homogeneous Linear Differential Equation :**

If RHS of Equation (1) is Equal to zero then we get homogeneous LDE.

ie $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + ...... + k_n y = 0$

Where $f(x)$ is the function of '$x$'

**The general solution format of Equation (1) of the form (C.S = C.F + P.I)**

Where C.S. – Complete Solution

C.F – Complementary function

P.I – Particular integral

So complete solution of Equation becomes (y = C.F + P.I)

***Note - 1:*** In case of Homogeneous LDE

C.S = C.F [where P.I = 0]

***Note - 2 :*** In case of Non-Homogeneous LDE

C.S = C.F + P.I

**Operator :**

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ .......by D, $D^2$, $D^3$ etc.

So that $\frac{dy}{dx} = Dy$

$\frac{d^2 y}{dx^2} = D^2 y$

................

$\frac{d^n y}{dx^n} = D^n y$

Where D–Derivative

Then $\frac{1}{D}$ – Integration

Then operator form of equation (1) becomes

$D^n y + K_1 D^{n-1} y + k_2 D^{n-2} y + ....... + K_n y = X$

$\Rightarrow$ $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + ........ + k_n) y = X$

$\Rightarrow$ F (D) y = X ........(2)

Where $F(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + ........ + k_n$ of function D

**Auxiliary Equation (AE)**

Putting the coefficient of y equal to Zero in Equation (2) we get an Auxiliary Equation. i.e. F (D) = 0

i.e. $D^n + k_1D^{n-1} + k_2D^{n-2} + ........ + k_n = 0$

Depending value of 'D' in Auxiliary Equation, complementary function are different types.

**Case - I :** If roots are real & Different

Let $m_1$ & $m_2$ are two real roots and different

i.e. $m_1 \neq m2$

Then C.F = $C_1em_1x + C_2em_2x$

Where $C_1$, $C_2$, arearbitrary constant

**Case - II :**If roots are real & Equal

Let $m_1$ & $m_2$ are two real roots & Equal

i.e $m_1 = m_2$

The C.F = $(C_1 + C_2x)\ em_1x$

Similarly if $m_1 = m_2 = m_3$ (Three roots are Equal)

Then C.F = $(C_1 + C_2x + C_3x^2)\, e^{m_1x}$

**Case - III :** If roots are Complex conjugate

Let $m_1 = \alpha \pm i\beta$ are conjugate complex root

Then C.F = $e^{\alpha x}\{C_1\cos\beta x + C_2\sin\beta x\}$

**Case - IV :** If two conjugate complex roots are equal

Let $m_1 = m_2 = \alpha \pm i\beta$ are equal

Then C.F = $e^{\alpha x}\{C_1 + C_2x)\cos\beta x + (C_3 + C_4x)\sin\beta x$

**Example – 1 :**

Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$ ......(1)

**Solution :**

The operator from of equation (1) becomes

$(D^2 - 8D + 15)\ y = 0$

So Auxiliary Equation

$D^2 - 8D + 15 = 0$

$\Rightarrow$ $(D - 3)(D - 5) = 0$

$\Rightarrow$ $D = 3, 5$

Then C.F = $C_1e^{3x} + C_2e^{5x}$

So complete Solution

$y = C_1e^{3x} + C_2e^{5x}$ (Ans)

**Example – 2 :**

Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

**Solution :**

The operator from of given equation is

$(D^2 - 6D + 9)\ y = 0$

Then A.E $D^2 - 6D + 9 = 0$

$\Rightarrow (D - 3)^2 = 0$

$\Rightarrow D = 3, 3$

C.F $= (C_1 + C_2x)\ e^{3x}$

Then C.S $y = (C_1 + C_2x)\ e^{3x}$ (Ans)

**Example – 3 :**

Solve $(D^2 + 4D + 5)\ y = 0$

**Solution :**

So A.E $D^2 + 4D + 5 = 0$

$$D = \frac{-4 \pm \sqrt{16 - 4.1.5}}{2.1}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

Then C.F $= e^{-2x}\ \{C_1 \text{Cos}x + C_2 \text{Sin}x\}$

So C.S $y = e^{-2x}\ \{C_1 \text{Cos}x + C_2 \text{Sin}x\}$ (Ans)

**Procedure to finding particular Integral.**

We know that F (D) y = X

$$\Rightarrow \quad y = \frac{X}{F(D)}$$

Depending upon nature of 'X', Particular integral are different types

**Case –1 :** When $X = e^{ax}$

Then P. I $= \frac{e^{ax}}{F(a)}$ where $D = a$

If $F(a) = 0$, Then PI $= \frac{xe^{ax}}{F'(a)}$ provided $F'(a) \neq 0$

If $F'(a) = 0$, Then PI $= \frac{x^2e^{ax}}{F''(a)}$ provided $F''(a) \neq 0$

And so on.

**Case – 2 :** Whex X = sin ($ax$ + b) or Cos ($ax$ + b)

Then PI $= \frac{\sin(ax+b)}{F(D^2)}$ Put $D^2 = -a^2$

But not D = – $a$

$= \frac{\text{Sin}(ax+b)}{F(-a^2)}$ provided F $(-a^2) \neq 0$

If F $(-a^2)$ = 0, The above rule Fails & We proceed further

ie P.I $= \frac{x\sin(ax+b)}{F'(-a^2)}$, Provided F′ $(-a^2) \neq 0$

If F′ $(-a^2)$ = 0, Then P.I $= x^2 \frac{\text{Sin}(ax+b)}{F''(-a^2)}$, Provided F″$(-a^2) \neq 0$

And so on

**Case – 3 :** When X = $e^{ax}$v, Where v = function of '$x$'

Then PI $= \frac{e^{ax}v}{F(D)}$

$= e^{ax} \frac{1}{F(D+a)} v$

Similary when X = $e^{-ax}$v

Then PI $= e^{-ax} \frac{1}{F(D+a)} v$

**Case – 4 :** When X = $x^m$ ($ie$, $x$, $x^2$, $x^3$ .........)

Then PI $= \frac{x^m}{F(D)}$ = $[F(D)]^{-1} x^m$

Convert F (D) into {1 + Φ(D)} or {1 – Φ(D)} by taking $D^m$
(if possible). Then by using Binomial Theorom we find solution.

**Case – 5 :** When X = xv

Then P.I $= \frac{xv}{F(D)}$

$= \left\{x - \frac{F'(D)}{F(D)}\right\} \frac{v}{F(D)}$ Where F′ (D) is the Derivative of F (D)

**Case – 6 :** When $x$ = is any other function

Then P.I $= \dfrac{x}{F(D)}$

Convert F (D) into (D – α) or (D + α) factor form

Then if $= \dfrac{x}{D-\alpha} = e^{ax}\int Xe^{-at}dx$ if $= \dfrac{x}{D-\alpha} = e^{-ax}\int Xe^{at}dx$

**Example – 4 :**

Find P. I of $(D^2 + 6D + 3)\, y = e^{2x}$

**Solution :**

P. I. $= \dfrac{e^{2x}}{D^2+6D+3}$ put D = a

i.e. D = 2

Then P.I. $= \dfrac{e^{2x}}{(2)^2+6(2)+3}$

$= \dfrac{e^{2x}}{4+12+3} = \dfrac{e^{2x}}{19}$ (Ans)

**Example – 5 :**

Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y\dfrac{dy}{dx} - 2y = e^x + \cos x$

**Solution :**

The operator form of given equation becomes

$(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$

So A.E $D^3 - 3D^2 + 4D - 2 = 0$

⇒ D – 1, 1 ± i

⇒ D = 1 , 1 ± i

C.F = $C_1 e^x + e^x \{C_2 \text{Cos}x + C_3 \text{Sin}x\}$

Then PI $= \dfrac{e^x + \cos x}{D^3 - 3D^2 + 4D - 2}$

$= \dfrac{e^x}{(D-1)(D^2-2D+2)} + \dfrac{\cos x}{D^3-3D^2+4D-2}$

$= \dfrac{e^x}{(D-1)\{1-2+2\}} + \dfrac{\cos x}{(-1)D-3(-1)+4D-2}$

$$= \frac{e^x}{D-1} + \frac{\cot x}{3D+1}$$

$$= x\frac{e^x}{1} + \frac{\cos x(3D-1)}{(3D+1)(3D-1)}$$

$$= xe^x + \frac{(3D)\cos x - \cos x}{9D^2 - 1}$$

$$= x\,e^x + \frac{-3\sin x - \cos x}{-9-1} = xe^x + \frac{1}{10}(3\sin x + \cos x)$$

So C.S y = $C_1 e^x + e^x \{C_2 \cos x + C^3 \sin x) + x\,e^x + \frac{1}{10}$ (3 sin$x$ + cos$x$}

**Example – 6 :**

Find the P.I. of ($D^3$ + 1) y = $e^x$ cos$x$ + sin3$x$

**Solution :**

$$\text{P.I.} = \frac{e^x \cos x + \sin 3x}{D^3 + 1}$$

$$= e^x \frac{\cos x}{(D+1)^3 + 1} + \frac{\sin 3x}{D^2 D + 1}$$

$$= e^x \frac{\cos x}{-D^3 + 3D^2 + 3D + 2} + \frac{\sin 3x}{-9.D + 1}$$

$$= e^x \frac{\cos x}{D^3 + 3D^2 + 3D + 2} + \frac{\sin 3x}{1 - 9D}$$

$$= e^x \frac{\cot x}{-D + 3(-1) + 3D + 2} + \frac{\sin 3x}{1 - 9D}$$

$$= e^x \frac{\cos x(2D+1)}{(2D-1)(2D+1)} + \frac{\sin 3x(1+9D)}{(1-9D)(1+9D)}$$

$$= e^x \frac{2D(\cos x) + \cos x}{4D^2 - 1} + \frac{\sin 3x + 9D(\sin 3x)}{1 - 81D^2}$$

$$= e^x \frac{-2\sin x + \cos x}{4(-1) - 1} + \frac{\sin 3x + 27\cos 3x}{1 - 81(-9)}$$

$$= \frac{e^x}{5}(2\sin x - \cos x) + \frac{1}{730}(\sin 3x + 27\cos 3x) \qquad \text{(Ans)}$$

**Example – 7 :**

Solve $\frac{d^2y}{dx^2}+9y=x\cos x$

**Solution :**

The operator form is $(D^2 + 9)\, y = x \cos x$

So A.E $D^2 + 9 = 0$

$\Rightarrow D^2 = -9$

$\Rightarrow D = \sqrt{-9}$

$\Rightarrow D = \pm 3i$

C.F = $C_1 \cos 3x + C_2 \sin 3x$

Now P.I $=\frac{x\cos x}{D^2+9}$

Here $F(D) = D^2 + 9$

$F'(D) = 2D$

Then PI $=\left\{x-\frac{F'(D)}{F(D)}\right\}\frac{V}{F(D)}$

$=\left\{x-\frac{2D}{D^2+9}\right\}\frac{\cos x}{D^2+9}$ put $D^2 = -1$

$=\left\{x-\frac{2D}{D^2+9}\right\}\frac{\cos x}{-1+9}$

$=\frac{x\cos x}{8}-\frac{2D(\cos x)}{8(D^2+9)}$

$=\frac{x\cos x}{8}+\frac{2\sin x}{8\times 8}$

$=\frac{x\cos x}{8}+\frac{\sin x}{32}=\frac{4x\cos x+\sin x}{32}$

So C.S $y = C_1\cos_3 x + C_2 \sin 3x + \frac{4x\cos x+\sin x}{32}$ (Ans)

**Example – 8 :**

Solve $\frac{d^2y}{dx^2}+4y=x^2$

**Solution :**

The operation form given equation becomes

$(D^2 + 4)\, y = x^2$

So A.E. $D_2 + 4 = 0$

$\Rightarrow$ $D^2 = -4$

$\Rightarrow$ $D = \sqrt{-4}$

$\Rightarrow$ $D = \pm 2i$

C.F = $C_1\cos 2x + C_2\sin 2x$

Then P.I $= \dfrac{x^2}{4\left(1+\dfrac{D^2}{4}\right)x^2}$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2$$

$$= \frac{1}{4}\left\{1-\frac{D^2}{4}+\frac{D^2}{16}\dots\dots\dots\right\}x^2 \text{ by using Binomial theorem}$$

$$= \frac{1}{4}\left\{x^2-\frac{D^2}{4}\left(x^2\right)+\frac{D^2}{16}\left(x^2\right)\dots\dots\dots\right\}$$

$$= \frac{1}{4}\left\{x^2-\frac{2}{4}+0\right\}$$

$$= \frac{1}{4}\left\{\frac{2x^2-1}{2}\right\} = \frac{2x^2-1}{8}$$

So C.S y = $C_1$ Cos$2x$ + $C_2$ sin $2x$ + $\dfrac{2x^2-1}{8}$ (Ans)

**Other Method for finding P. I :**

Method of variation of Parameters :

This method is applies to equations of the form

$y'' + py' + qy = x$

Where p, q & $x$ are function of $x$.

Then P. I = $\boxed{-y_1\int \frac{y_2 x}{w}dx + y_2\int \frac{y_1 x}{w}dx}$

Where $y_1$ & $y_2$ are the solution of $y'' + py' + qy = 0$ of the form $= c_1y_1 + c_2y_2$ & w is called wronskian of $y_1$ & $y_2$

Calculate by formula w $(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix}$

**Example – 9 :**

Solve $\frac{d^2y}{dx^2} + y = \text{cosec}x$

**Solution :**

The operator form of given equation is

$(D^2 + 1) = \text{Cosec}x$

So A.E $D^2 + 1 = 0$

$\Rightarrow D^2 = -1$

$\Rightarrow D = \sqrt{-1} = 0 \pm i$

C.F. = $C_1 \cos x + C_2 \sin x$

Here $y_1 = \cos x$ $y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

Then P.I $= -\cos x \int \frac{\sin x.\text{cosec}x}{1} dx + \sin x \int \frac{\sin x.\text{cosec}x}{1} dx$

$$= -\cos x \int \sin x . \frac{1}{\sin x} dx + \sin x \int \cos x \frac{1}{\sin x} dx$$

$$= -\cos x \int dx + \sin x \int \cot x \, dx$$

$$= -\cos x\,(x) + \sin x \ln \sin x$$

So C.S $y = C_1 \cos x + C_2 \sin x + \sin x \text{ Insin}x - x \cos x$ (Ans.)

**Partial Differential Equation**

Let $z = f(x, y)$ be a function containing two independent variable $x$ & y and z is the Dependent variable.

**Notation :** Let $z = f(x, y)$ be a function of $x$ & y

Then $\frac{\partial z}{\partial x} = p$ $\quad \frac{\partial z}{\partial y} = q$

$$\frac{\partial^2 z}{\partial x^2} = r \qquad \frac{\partial^2 z}{\partial y^2} = t$$

$$\frac{\partial^2 z}{\partial x \partial y} = S$$

**Formation of Partial differential Equation**

A partial differential equation can be formed by

(i) Eliminating arbitrary constant.

(ii) Eliminating arbitrary function.

**Example – 10 :**

Form a partial different equation by eliminating function

$Z = f(x^2 + y^2)$ ...(1)

**Solution :**

Differentiating partially w.r.t. $x$ & y in equation (1) we get

$$\frac{\partial z}{\partial x} = f'\left(x^2 + y^2\right).2x \text{ (taking y as a constant)}$$

$\Rightarrow$ $p = f'(x^2 + y^2).\ 2x$ ...(2)

Similarly $q = f'(x^2 + y^2).\ 2y$ ...(3)

Dividing (2) & (3) we get $\dfrac{p}{q} = \dfrac{f'\left(x^2 + y^2\right).2x}{f'\left(x^2 + y^2\right).2y}$

$$\frac{p}{q} = \frac{x}{y}$$

$\Rightarrow$ $py - qx = 0$ (Ans.)

**Linear Equation of the First order :**

A Linear partial differential equation of the 1st order is of the form

$Pp + Qq = R$

Where P, Q & R are function of $x$, y, z.

This equation also known as Lagrange's Linear equation

**NOTE :**

The general solution of the liner partial differential equation $Pp + Qq = R$ is

$\phi(a, b) = 0$

Or $a = \phi(b)$

Or $b = \phi(a)$

Where $\phi$ is an arbitrary function & $u(x, y, z) = a$ & $v(x, y, z) = b$ form the solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Then that can be solved by two methods

(1) Method for Grouping

(2) Method for Multipliers

**Method or grouping :**

Take any two fraction from Subsidiary Equation such that the 3rd variable is absent or it may be cancelled.

For example take $\dfrac{dx}{P} = \dfrac{dy}{Q}$ (such that z may be absent)

After Integration we get $f(x, y) = a$

Similarly we take $\frac{dy}{Q} = \frac{dz}{R}$

After Integation $f(y, z) = b$

So general solution is $a = \phi(b)$

or $b = \phi(a)$

or $\phi(a, b) = 0$

**Method for Multipliers**

Let us choose the multiplier's $(P', Q', R')$ such

That $PP' + QQ' + RR' = 0$

Then we write $P'dx + Q'dy + R'dz = 0$

On Integration we get $f(x, y, z) = a$

Similarly choosing the multipliers $(P'', Q'', R'')$ such that

$PP'' + QQ'' + RR'' = 0$

On Integration we get $g(x, y, z) = b$

So general solution is $a = \phi(b)$ or $\phi(a, b) = 0$

**Example – 11 :**

Solve $y^2zp + z^2xq = y^2x$

**Solution :**

It is of the form $Pp + Qq = R$

Where $P = y^2z$, $Q = z^2x$, $R = y^2x$

So its S.E $\frac{dx}{y^2z} = \frac{dy}{z^2x} = \frac{dz}{y^2x}$

Taking 1st and 3rd fraction, we get

$\frac{dx}{y^2z} = \frac{dy}{y^2x}$ (Here 3rd variable $y^2$ is cancelled)

$\Rightarrow$ $xdx = zdx$

Integrating both sides we get $\int xdx = \int zdz$

$\Rightarrow$ $\frac{x^2}{2} = \frac{z^2}{2} + c$

$\Rightarrow$ $x^2 - z^2 = 2c = a$

Similarly taking 2nd and 3rd

i.e. $\frac{dy}{z^2x} = \frac{dz}{y^2x}$

$\Rightarrow$ $y^2dy = z^2dz$

Integrating both sides we get

$\Rightarrow \quad \frac{y^3}{3} = \frac{z^3}{3} + c^1$

$\Rightarrow \quad y^3 - z^3 = 3c^1 = b$

So general solution in $x^2 - z^2 = \phi\,(y^3 - z^3)$ (Ans.)

**Example – 12 :**

Solve $x\,(z^2 - y^2)\,p + y\,(x^2 + y^2)\,q = z = (y^2 - x)$

**Solution :**

It is the equation of the form

$Pp + Qq = R$

Where $P = x\,(z^2 - y^2)$ $Q = y\,(x^2 - z^2)$ $R = z\,(y^2 - x^2)$

So its S.E is $\frac{dx}{x\left(z^2 - y^2\right)} = \frac{dy}{y\left(x^2 - z^2\right)} = \frac{dz}{z\left(y^2 - x^2\right)}$

Let us choose multipliers $(x, y, z)$ *i.e* $P' = x$, $Q' = y$, $R' = z$

Such that $x.x\,(z^2 - y^2) + y.y\,(x^2 - z^2) + z.z\,(y^2 - x^2)$

$= x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + z^2y^2 - z^2x^2$

$= 0$

Then we write $xdx + ydy + zdz = 0$

On integration we set

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$

$\Rightarrow \quad x^2 + y^2 + z^2 = 2c = a$

Again choose the multipliers $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ ie $P'' = \frac{1}{x}$, $Q'' = \frac{1}{y}$, $R'' = \frac{1}{z}$

Such that $\frac{1}{x}x\left(z^2 - y^2\right) + \frac{1}{y}y\left(x^2 - z^2\right) + \frac{1}{z}z\left(y^2 - x^2\right)$

$= z^2 - y^2 + x^2 - z^2 + y^2 - x^2 = 0$

Then $\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$

On integration we get

$\log x + \log y + \log z = \log b$

$\Rightarrow \quad \log\,(xyz) = \log b$

$\Rightarrow \quad xyz = b$

So general solution in $x^2 + y^2 + z^2 = \phi\,(xyz)$ (Ans)

## *Assignment*

**Solve the followings :**

1. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-10y=6e^{4x}$
2. $y''+3y'+2y=4\cos^2 x$
3. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2e^x$
4. $(D^2+a^2)y=k\cos(ax+b)$
5. $(D-2)^2y=8(e^{2x}+\sin 2x)$
6. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=1+x^2$

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# LAPLACE TRANSFORMS

**GAMMA FUNCTION :**

The grmma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0 \qquad ...(1)$$

It defines a function of n for positive values of n.

**Value of Γ (1) :**

We have,

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x} dx = \left| -e^{-x} \right|_0^\infty = 1$$

Hence, $\Gamma(1) = 1$ ...(2)

**<u>Reduction formula for Γ (n)</u>** :

We have,

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx \text{ [Integrating by parts]}$$

$$= \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx = 0 + n\Gamma(n)$$

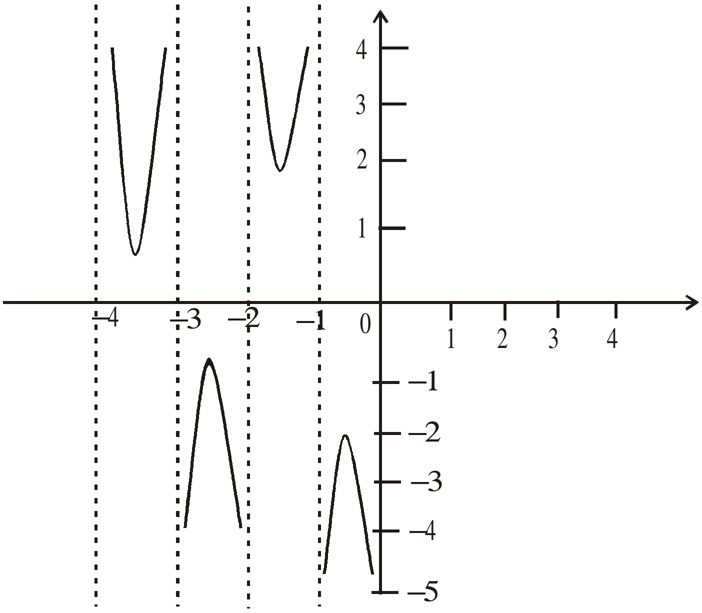
$\therefore \Gamma(n+1) = n\,\Gamma(n)$, ....(3)

which is the reducation formula for Γ(n).

Using the reduction formula for Γ(n), we can write the value of Γ(n) in the form,

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}. \qquad ....(4)$$

Thus (1) and (4) together give a complete definition of Γ(n) defined for all values of n except when n is zero or a negative integer and its graph is as shown in the following figure.




**VALUE OF Γ(n) IN TERMS OF FACTORIAL**

Using $\Gamma(n + 1) = n\Gamma(n)$ successively, we get

$\Gamma(2) = \Gamma(1 + 1) = 1 \times \Gamma(1) = 1!$

$\Gamma(3) = \Gamma(2 + 1) = 2 \times \Gamma(2) = 2 \times 1 = 2!$

$\Gamma(4) = \Gamma(3 + 1) = 3 \times \Gamma(3) = 3 \times 2! = 3!$

-------------------------------------------------

In general $\Gamma(n + 1) = n!$., provided n is a positive integer.

Taking n = 0, it defines $0! = \Gamma(1) = 1$

Thus, $\Gamma(n + 1) = n!$ (for n = 0, 1, 2, 3......) ....(5)

**<u>Value of $\Gamma\frac{1}{2}$</u>:**

We have,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2} dx \quad \text{[Put } x = y_2 \text{ so that } dx = 2ydy]$$

$$= 2\int_0^\infty e^{-y^2} dy \text{, Which is also } = 2\int_0^\infty e^{-x^2} dx$$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy \quad \text{[Put } x = r\cos\theta \text{ and } y = r\sin\theta]$$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr\, d\theta = 4 \cdot \frac{\pi}{2} \int_0^\infty e^{-r^2} r dr = 2\pi = \left[\left(-\frac{1}{2} e^{-r^2}\right)\right]_0^\infty = \pi$$

Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = 1.772$ ....(6)

**Example – 1 :**

Evaluate, $\dfrac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(4).\Gamma(6).\Gamma(8)}$

**Solution :**

We have, $\dfrac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(4).\Gamma(6).\Gamma(8)}$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)\cdot\Gamma\left(\frac{3}{2}+1\right)\cdot\sqrt{\pi}}{\Gamma(3+1).\Gamma(5+1).\Gamma(7+1)} = \frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)\cdot\sqrt{\pi}}{3!.5!.7!}$$

$$= \frac{\sqrt{\pi}.3.\Gamma\left(\frac{1}{2}+1\right).\sqrt{\pi}}{4.3!.5!.7!} = \frac{\frac{3}{2}\pi.\Gamma\left(\frac{1}{2}\right)}{4.3!.5!.7!} = \frac{3\pi\sqrt{\pi}}{8.3!.5!.7!}$$

$$= \frac{\pi\sqrt{\pi}}{16!.5!.7!} = \frac{\pi\sqrt{\pi}}{9676800}$$

**Example – 2 :**

Evaluate $\Gamma(-3.5)$

**Solution :**

We know that, $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

For all n expect n is zero or a negative integer.

Now, we have,

$$\Gamma(-3.5) = \frac{\Gamma(-3.5+1)}{-3.5} = \frac{\Gamma(-2.5)}{-3.5} = \frac{\Gamma(-2.5+1)}{(-3.5)\,(-2.5)} = \frac{\Gamma(-1.5)}{(3.5)\,(2.5)}$$

$$= \frac{\Gamma(-1.5+1)}{(3.5)\,(2.5)(-1.5)} = \frac{\Gamma(-0.5)}{(3.5)\,(2.5)(-1.5)} = \frac{\Gamma(-0.5+1)}{(3.5)\,(2.5)(-1.5)(-0.5)}$$

$$= \frac{\Gamma(0.5)}{(3.5)\,(2.5)(1.5)(0.5)} = \frac{\sqrt{\pi}}{(3.5)\,(2.5)(1.5)(0.5)} = 0.27$$

$\therefore\ \Gamma(-3.5) = 0.27$

**Laplace transforms :**

**Definition :**

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $f$(t), denoted by L {$f$(t)} is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Provided that the integral exists. s is a parameter which may be a real or complex number.

L{$f$(t)} being clearly a function of s is briefly written as $\bar{f}(s)$.

i.e., L{$f$(t)} = $\bar{f}(s)$.

This implies that, $f(t) = L^{-1}\{\bar{f}(s)\}$

Then $f$(t) is called the inverse Laplace transform of $\bar{f}(s)$.

The symbol L, which transforms f(t) into $\bar{f}(s)$, is called the Laplace transformation operator.

**CONDITIONS FOR THE EXISTENCE :**

The Laplace transform of $f$(t) i.e., $\int_0^\infty e^{-st} f(t)dt$ exists for s > a, if

(i) $f$ (t) is continuous

and (ii) $\lim_{t\to\infty} \{e^{-at} f(t)\}$ is finite.

**TRANSFORMS OF ELEMENTARY FUNCTIONS :**

The direct application of the definition gives the following formulae :

(1) $L\{1\} = \frac{1}{s}$ (s > 0)

(2) $L\{t^n\} = \begin{cases} \frac{n!}{s^{n+1}}, \text{ when } n = 0,1,2,3,\ldots\ldots\ldots \\ \frac{\Gamma(n+1)}{s^{n+1}}, \text{ otherwise } (s > 0) \end{cases}$

(3) $L\{e^{at}\} = \frac{1}{s-a}$ (s > $a$)

(4) $L\{\sin at\} = \frac{a}{s^2+a^2}$ (s > 0)

(5) $L\{\cos at\} = \frac{s}{s^2-a^2}$ (s > 0)

(6) $L\{\sin h\, at\} = \frac{a}{s^2-a^2}$ (s > $|a|$)

(7) L {cosh $at$} $= \dfrac{s}{s^2 - a^2}$ (s > |$a$|)

**PROOFS :**

(1) $L\{1\} = \int_0^\infty e^{-st}.1.dt = \left|\dfrac{-e^{-st}}{s}\right|_0^\infty = \dfrac{1}{s}$, if s > 0

(2) $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \int_0^\infty e^{-p}.\left(\dfrac{p}{s}\right)^n.\dfrac{dp}{s}$, on putting st = p

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p}.p^n dp$$

$= \dfrac{\Gamma(n+1)}{s^{n+1}}$, if n > –1 and s > 0

If n is a positive integer, Γ(n + 1) = n!. Therefore, $L\{t^n\} = \dfrac{n!}{s^{n+1}}$, if s > 0

(3) $L\{e^{at}\} = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = dt = \left|\dfrac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \dfrac{1}{s-a}$, if s > $a$

(4) L {sin $at$} = $\int_0^\infty e^{-st}\sin atdt = \left|\dfrac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right|_0^\infty = \dfrac{a}{s^2+a^2}$, if s > 0

(5) L {cos $at$} $= \int_0^\infty e^{-st}\cos at\ dt = \left|\dfrac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right|_0^\infty = \dfrac{s}{s^2+a^2}$, if s > 0

(6) L {sin h at} = $\int_0^\infty e^{-st}\sinh at\ dt = \int_0^\infty e^{-st}\left(\dfrac{e^{at}-e^{-at}}{2}\right)dt$

$= \dfrac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)t}dt\right] = \dfrac{1}{2}\left[\dfrac{1}{s-a} - \dfrac{1}{s+a}\right] = \dfrac{a}{s^2-a^2}$, for s > |$a$|

(7) L{cosh $at$} = $\int_0^\infty e^{-st}\cosh atdt = \int_0^\infty e^{-st}\left(\dfrac{e^{at}+e^{-at}}{2}\right)dt = \dfrac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)t}dt\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{s-a} + \dfrac{1}{s+a}\right] = \dfrac{s}{s^2-a^2}$, for s > |$a$|

**PROPERTIES OF LAPLACE TRANSFORMS :**

**1. LINEARITY PROPERTY :**

If $a$, b, c be any constants and f, g, h any functions of t, then

L{$af$ (t) + bg(t) – ch(t) } = $a$L {$f$(t)} + bL {g(t)} – cL {h(t)}

By definition,

$$\text{L.H.S} = \int_0^\infty e^{-st}[af(t)+bg(t)-ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt - c\int_0^\infty e^{-st}h(t)dt$$

= $a$L {$f$(t)} + bL {g(t)} – cL{h(t)}

**II. FIRST SHIFTING PROPERTY :**

If L {$f$(t)} = $\bar{f}(s)$, then

L {$e^{at} f$(t)} = $\bar{f}(s-a)$

By definition,

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt$$

$= \int_0^\infty e^{-rt}f(t)dt$, where r = s – $a$ $= \bar{f}(r) = \bar{f}(s-a)$.

**APPLICATION OF FIRST SHIFTING PROPERTY :**

(1) $L\{e^{at}\} = \dfrac{1}{s-a}$

(2) $L\{e^{at}\, t^n\} = \dfrac{n!}{(s-a)^{n-1}}$ when n = 1, 2, 3,.........

(3) $L\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

(4) $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2-b^2}$

(5) $L\{e^{at} \sin h\, bt\} = \dfrac{b}{(s-a)^2-b^2}$

(6) $L\{e^{at} \cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$

where in each case > $a$.

**III. CHANGE OF SCALE PROPERTY :**

If L $\{f(t)\} = \bar{f}(s)$, then

L $\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

By definition,

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

$$= \int_0^\infty e^{-su/a} f(u) \frac{du}{a} \qquad \text{[put } at = u \text{ dt} = \frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

**Example –3 :**

Find the Laplace transform of $e^{2t}$ $(3t^5 - \cos 4t)$.

**Solution :**

L$\{e^{2t}(3t^5 - \cos 4t)\}$

$= 3$L$\{e^{2t}t^5\} -$ L$\{e^{2t}\cos 4t\}$

$$= 3 \cdot \frac{5!}{(s-2)^6} - \frac{s-2}{(s-2)^2 + 4^2} = \frac{360}{(s-2)^6} - \frac{s-2}{s^2 - 4s + 20}$$

**Example – 4 :**

Find the laplace transform of $e^{-t}\sin^2 3t$

**Solution :**

We have

$$L\{\sin^2 3t\} = \frac{1}{2}\text{ L } \{1 - \cos 6t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 6^2}\right] = \frac{18}{s(s^2 + 36)} = \bar{f}(s)$$

**Example – 5 :**

Find the laplace transform of $e^{-3t}$ sin 5t sin 3t.

**Solution :**

We have, L{sin 5t sin 3t} $= \frac{1}{2}$ L{cos2t – cos8t}

$$= \frac{1}{2}\left[\frac{s}{s^2 + 2^2} - \frac{s}{s^2 + 8^2}\right] = \frac{30s}{(s^2 + 4)(s^2 + 64)} = \bar{f}(s).$$

By first shifting property, we get

$$L\{e^{-3t}\sin 5t \sin 3t\} = \bar{f}(s+3$$

$$= \frac{30(s+3)}{\{(s+3)^2+4\}\{(s+3)^2+64\}} = \frac{30(s+3)}{\{s^2+6s+13\}\{s^2+6s+73\}}$$

**Example – 6 :**

Find the laplace transform of $e^{-2t}$ $(2\sqrt{t}-3/\sqrt{t})$

**Solution :**

We have

$$L\{2\sqrt{t}-3/\sqrt{t}\} = 2L\left\{t^{\frac{1}{2}}\right\} - 3L\left\{t^{\frac{-1}{2}}\right\} = 2\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} - 3\frac{\Gamma\left(\frac{-1}{2}+1\right)}{s^{\frac{-1}{2}+1}}$$

$$= \frac{2\Gamma\left(\frac{1}{2}\right)}{2s^{\frac{3}{2}}} - 3\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} - 3\frac{\sqrt{\pi}}{\sqrt{s}} = \bar{f}(s).$$

By first shifting property, we get

$$L\{e^{-2t}\left(2\sqrt{t}-3/\sqrt{t}\right) = \bar{f}(s+2)$$

$$= \frac{\sqrt{\pi}}{(s+2)^{\frac{3}{2}}} - \frac{3\sqrt{\pi}}{\sqrt{s+2}} = \frac{\sqrt{\pi}}{(s+2)\sqrt{s+2}} - \frac{3\sqrt{\pi}}{\sqrt{s+2}}$$

**Example – 7 :**

Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$

**Solution :**

Given that, $L\left\{\frac{\sin at}{t}\right\} = L\{f(t)\} = \tan^{-1}\left(\frac{1}{s}\right) = \bar{f}(s)$

By change of scale property, we get

$$L\{f(at)\} = L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$$

$$\therefore\ \frac{1}{a}L\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right)$$

Therefore, $L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$

**LAPLACE TRANSFORMS OF DERIVATIVES :**

(1) $f'(t)$ be continuous and $L\{f(t)\} = \overline{f}(s)$, then $L\{f'(t))\} = s\overline{f}(s) - f(0)$.

**Proof :** We have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= | e^{-st} f(t) |_0^\infty - \int_0^\infty (-s)e^{-st}.f(t)dt$$

Now assuming f(t) be such that $\lim_{t\to\infty} e^{-st} f(t) = 0$, we have

$$L\{f'(t)\} = -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

Thus, $L\{f'(t)\} = s\,\overline{f} - f(0)$

(2) If $f'$ (t) and its first (n – 1) derivatives be continuous, then

$$L\{f^n(t)\} = s^n\ \overline{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ........ - f^{n-1}(0)$$

Thus,

$$L\{f''(t)\} = s^2\overline{f}(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3\overline{f}(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$L\{f^{iv}(t)\} = s^4\overline{f}(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$$

and so on.

**Laplace transforms of integrals :**

If $L\{f(t)\} = \overline{f}(s), then\ L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\overline{f}(s)$

**Proof :**

Let $\phi(t) = \int_0^t f(u)du$, then $\phi'(t) = f(t)$ and $\phi(0) = 0$

$\therefore$ $L\{\phi'(t)\} = s\overline{\phi}(s) - \phi(0)$

or $L\{f(t)\} = s\overline{\phi}(s)$

or $\overline{f}(s) = s\overline{\phi}(s)$

or $\overline{\phi}(s) = \frac{1}{s}\overline{f}(s)$

Hence, $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\overline{f}(s)$.

**Multiplication By $t^n$ :**

If $L\{f(t)\} = \bar{f}(s)$, then

$L\{t^n f(t)\} = (-1)^n \frac{s^n}{ds^n} \left[\bar{f}(s)\right]$, where n = 1, 2, 3............

**Division By t :**

If $L\{f(t)\} = \bar{f}(s)$, then

$L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} \bar{f}(s)ds$ , provided the integral exists.

**Example –8 :**

Find the laplace transforms of

(1) t sin at

(2) t cos at

**Solution :**

(1) We have, $L\{\sin at\} = \frac{a}{s^2 + a^2}$

$\therefore L\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) = -\left[\frac{-2as}{(s^2 + a^2)^2}\right]$

Hence $\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}$

(2) We have, $L\{\cos at\} = \frac{s}{s^2 + a^2}$

$\therefore L\{t \cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right]$

Hence, $L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$

**Example – 9 :**

Find the laplace transforms of $t^2$ cos at.

**Solution :**

We have, $L\{\cos at\} = \frac{s}{s^2 + a^2}$

$\therefore L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right]$

$$= \frac{d}{ds}\left[\frac{a^2 - s^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{-2s(s^2 + a^2)^2 - 2(a^2 - s^2).2s(s^2 + a^2)}{(s^2 + a^2)^4}$$

$$= \frac{-2s(s^2 + a^2) + 4s(s^2 - a^2)}{(s^2 + a^2)^3} = \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}$$

**Example – 10 :**

Find the laplace transforms of $\frac{(e^{-at} - e^{-bt})}{t}$.

**Solution :**

We have, L $\{e^{-at} - e^{-bt}\} = \frac{1}{(s+a)} - \frac{1}{(s+b)}$

$$\therefore\ L\left\{\frac{(e^{-at} - e^{-bt})}{t}\right\} = \int_s^\infty \left[\frac{1}{(s+a)} - \frac{1}{(s+b)}\right] ds$$

$$= \left[\log\,(s+a) - \log(s+b)\right]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)\Big]_s^\infty = \log\frac{s(1+a/s)}{s(1+b/s)}\Big]_s^\infty$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+b}{s+a}\right)$$

**Example – 11 :**

Find the inverse Laplace transform of $\left(\frac{e^{at} - \cos bt}{t}\right)$

**Solution :**

We have, L $\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_s^\infty \left[\frac{1}{s-a} - \frac{s}{s^2 + b^2}\right] ds$

$$\therefore\quad L\left\{\frac{\left(e^{at} - \cosh t\right)}{t}\right\} = \int_s^\infty \left[\frac{1}{s-a} - \frac{s}{s^2 + b^2}\right] ds$$

$$= \left[ \log(s-a) - \frac{1}{2}\log\left(s^2+b^2\right) \right]_s^{\infty}$$

$$= \frac{1}{2}\left[ 2\log(s-a) - \log\left(s^2+b^2\right) \right]_s^{\infty}$$

$$= \frac{1}{2}\log\left( \frac{s-a}{s^2+b^2} \right) \Bigg]_s^{\infty}$$

$$= \frac{1}{2}\left[ \log 1 - \log\left( \frac{(s-a)^2}{s^2+b^2} \right) \right]$$

$$= -\frac{1}{2}\log\left( \frac{(s-a)^2}{s^2+b^2} \right)$$

$$= \frac{1}{2}\log\left[ \frac{s^2+b^2}{(s+b)^2} \right].$$

**INVERSE LAPLACE TRANSFORMS :**

We know that if $\{f(t)\} = \overline{f}(s)$, then $L^{-1}\{\overline{f}(s)\} = f(t)$

Let us now determine the inverse Laplace transforms of some given function of s.

(1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

(2) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(3) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, \ldots\ldots\ldots\ldots$

(4) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!}, n = 1, 2, 3, \ldots\ldots\ldots\ldots$

(5) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

(6) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \cos at$

(7) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

(8) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at$

(9) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=\frac{1}{b}e^{at}\sin bt$

(10) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$

(11) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{1}{2a}t\sin at$

(12) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}(\sin at-at\cos at)$

**INVERSE LAPLACE TRANSFORMS BY THE METHOD OF PARTIAL FRACTIONS :**

We have seen that $L\{f(t)\}$ in many cases, is a rational algebraic function of s. Hence to find the inverse laplace transforms of $\bar{f}(s)$, we first express the given function of s into partial fractions which will, then, be recognizable as one of the above mentioned standard forms.

**Example – 12:**

Find the inverse laplace transform of $\frac{s^2+s+2}{(s+1)^2(s-3)}$.

**Solution :**

Suppose that,

$$\frac{s^2+s+2}{(s+1)^2(s-3)}=\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{(s-3)} \qquad ....(1)$$

Multiplying both sides of (1) by $(s + 1)^2 (s – 3)$, we get

$s^2 + s + 2 = A (s + 1) (s – 3) + B (s – 3) + C(s + 1)^2$ ...(2)

Putting $s = –1$

$$2 = -4B \Rightarrow B = -\frac{1}{2}$$

Putting $s = 3$

$$14 = 16C \Rightarrow C = \frac{7}{8}$$

Equating co-efficient of $s^2$, we get

$$1 = A + C \Rightarrow A = 1 - C \Rightarrow A = \frac{1}{8}$$

Putting the values of A, B, C in (1) we get

$$\frac{s^2+s+2}{(s+1)^2(s-3)} = \frac{1}{8}\cdot\frac{1}{(s+1)} - \frac{1}{2}\cdot\frac{1}{(s+1)^2} + \frac{7}{8}\cdot\frac{1}{(s-3)}$$

$$\therefore\ L^{-1}\left\{\frac{s^2+s+2}{(s+1)^2(s-3)}\right\}$$

$$= \frac{1}{8}L^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{7}{8}L^{-1}\left\{\frac{1}{(s-3)}\right\}$$

$$= \frac{1}{8}e^{-t} - \frac{1}{2}e^{-t}.t + \frac{7}{8}e^{3t}$$

**Example – 13 :**

Find the inverse laplace transforms of $\frac{s}{(s-2)\,(s^2+9)}$

**Solution :**

Suppose that,

$$\frac{s}{(s-2)\,(s^2+9)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+9} \qquad ...(1)$$

Multiplying both sides by (s – 2) ($s^2$ + 9), we get

S = A ($s^2$ + 9) + (Bs + C) (s – 2) ...(2)

Putting s = 2, 2 = 13A ⇒ A = $\frac{2}{13}$

Putting s = 0, 0 = 9A – 2c ⇒ C = $\frac{9}{13}$

Equating co-efficient of $s^2$, we get

0 = A + B ⇒ B = $-\frac{2}{13}$

Putting the values of A, B, C in (1), we get

$$\frac{s}{(s-2)\,(s^2+9)} = \frac{2}{13}\cdot\frac{1}{s-2} - \frac{2}{13}\cdot\frac{s}{(s^2+9)} + \frac{9}{13}\cdot\frac{1}{(s^2+9)}$$

$$\therefore\ L^{-1}\left\{\frac{s}{(s-2)\,(s^2+9)}\right\}$$

$$= \frac{2}{13}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{13}L^{-1}\left\{\frac{s}{(s^2+9)}\right\} + \frac{9}{13}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{2}{13}e^{2t} - \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t\,.$$

**OTHER METHODS OF FINDING INVERSE LAPLACE TRANSFORMS :**

**(I) SHIFTING PROPERTY :**

If $L^{-1}\{\bar{f}(x)\} = f(t)$, then

$L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t) = e^{at} L^{-1}\{\bar{f}(s)\}$

**(II)** If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$, then $L^{-1}\{s.\bar{f}(s)\} = \frac{d}{dt} f(t)$.

In general, $L^{-1}\{s^n \bar{f}(s)\} = \frac{d^n}{dt^n}\{f(t)\}$,

Provided $f(0) = f'(0) = \ldots \ldots \ldots = f^{n-1}(0) = 0$.

**(III)** If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$

**(IV)** If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $t.f(t) = L^{-1}\left\{-\frac{d}{ds}\{\bar{f}(s)\}\right\}$

**(V)** If $f(t) = L^{-1}\{\bar{f}(s)\}$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$.

This formula is useful in finding $f(t)$ when $\bar{f}(s)$ is given.

**Example – 14 :**

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s}\right)$.

**Solution :**

Let $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s}\right)\right\} = f(t)$

$\Rightarrow \quad L\{f(t)\} = \tan^{-1}\left(\frac{2}{s}\right) = \bar{f}(s)$

Then by formula IV we get,

$$L\{t.f(t)\} = -\frac{d}{ds}\left[\bar{f}(s)\right] = -\frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s}\right)\right]$$

$$= -\left[\frac{1}{1+(2/s)^2}\right]\cdot\left[\frac{-2}{s^2}\right]$$

$$= \frac{2}{s^2+4}$$

$\Rightarrow \quad tf(t) = L^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

$\Rightarrow \quad f(t) = \frac{\sin 2t}{t}$

$\therefore \quad L^{-1}\left\{\tan^{-1}\left(\frac{2}{s}\right)\right\} = \frac{\sin 2t}{t}$ .

**Example – 16 :**

Find the inverse Laplace transform of log $\left(\frac{s}{s+1}\right)$.

**Solution :**

Let $L^{-1}\left\{\log\left(\frac{s}{s+1}\right)\right\} = f(t)$

$\Rightarrow \quad L\{f(t)\} = \log\left(\frac{s}{s+1}\right) = \bar{f}(s)$

Then by formula IV we get,

$$L\{t.f(t)\} = -\frac{d}{ds}\left[\log\left(\frac{s}{s+1}\right)\right] = \frac{d}{ds}\left[\log s - \log(s+1)\right]$$

$$= -\left[\frac{1}{s} - \frac{1}{s+1}\right] = \frac{1}{s+1} - \frac{1}{s}$$

$\Rightarrow \quad t.f(t) = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s}\right\} = e^{-t} - 1$

$\Rightarrow \quad f(t) = \frac{e^{-t}-1}{t}$

$\therefore \quad L^{-1}\left\{\log\left(\frac{s}{s+1}\right)\right\} = \frac{\left(e^{-1}-1\right)}{t}$

**Example – 17 :**

Find the inverse Laplace transform of $\frac{1}{s^2\left(s^2+a^2\right)}$.

**Solution :**

We have

$L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\right\} = \frac{1}{a}$ sin at $= f(t)$.

Then by Formula III we get,

$$L^{-1}\left\{\frac{1}{s\left(s^2+a^2\right)}\right\}=\frac{1}{a}\int_0^t \sin at\, dt=-\frac{1}{a^2}\cos at\Big]_0^t \frac{1-\cos at}{a^2}$$

Thus we have,

$$L^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\}=\frac{1}{a^2}\int_0^t\left(1-\cos at\right)dt$$

$$=\frac{1}{a^2}\left[t-\frac{1}{a}\sin at\right]_0^t$$

$$=\frac{1}{a^2}\left[t-\frac{1}{a}\sin at\right]_0^t$$

$$=\frac{1}{a^3}\left(at-\sin at\right).$$

**Example – 18 :**

Find the inverse Laplace transform of $\frac{1}{s^2\left(s+5\right)}$.

**Solution :**

We have, $L^{-1}\left\{\frac{1}{s+5}\right\}=e^{-5t}=f\left(t\right)$.

Then by Formula III, we get

$$L^{-1}\left\{\frac{1}{s\left(s+5\right)}\right\}=\frac{1}{5}\int_0^t\left[1-e^{-5t}\right]dt$$

$$L^{-1}\left\{\frac{1}{s\left(s+5\right)}\right\}=\int_0^t e^{-5t}dt=-\frac{1}{5}e^{-5t}\Big]_0^t=-\frac{1}{5}\left[e^{-5t}-1\right]=\frac{1}{5}\left[1-e^{-5t}\right]$$

Thus we have,

$$L^{-1}\left\{\frac{1}{s^2\left(s+5\right)}\right\}=\frac{1}{5}\int_0^t\left[1-e^{-5t}\right]dt$$

$$=\frac{1}{5}\left[t+\frac{1}{5}e^{-5t}\right]dt$$

$$=\frac{1}{5}\left[t+\frac{1}{5}e^{-5t}\right]_0^t$$

$$=\frac{1}{5}\left[t+\frac{1}{5}e^{-5t}-\frac{1}{5}\right]$$

$$=\frac{1}{25}\left[e^{-5t}+5t-1\right]$$

**Example – 19 :**

Find the inverse Laplace transform of $\frac{s^2}{\left(s^2+a^2\right)^2}$

**Solution :**

We have,

$$L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)^2}\right\}=\frac{1}{2a}t\sin at=f\left(t\right)$$

Since $f(0) = 0$, we get from Formula II that,

$$L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)^2}\right\}=L^{-1}\left\{s.\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{d}{dt}\left[f\left(t\right)\right]$$

$$=\frac{d}{dt}\left[\frac{1}{2a}t\sin at\right]$$

$$=\frac{1}{2a}(\sin at + at\cos at).$$

**Example –20 :**

Find the inverse Laplace transform of $\frac{s+3}{\left(s^2+6s+13\right)^2}$ .

**Solution :**

We have, $\frac{s+3}{\left(s^2+6s+13\right)^2}=\frac{s+3}{\left[\left(s^2+6s+9\right)+4\right]^2}=\frac{s+3}{\left[\left(s+3\right)^2+4\right]^2}$

Then by formula I we get

$$L^{-1}\left\{\frac{s+3}{\left(s^2+6s+13\right)^2}\right\}=L^{-1}\left\{\frac{s+3}{\left(\left(s+3\right)^2+2^2\right)^2}\right\}=e^{-3t}.L^{-1}\left\{\frac{s}{\left(s^2+2^2\right)^2}\right\}$$

$$=\frac{1}{4}e^{-3t}.\ t\sin 2t.$$

**Example – 21 :**

Find the inverse Laplace transform of $\dfrac{s}{\left(s^2+a^2\right)^2}$.

**Solution :**

Let, $f(t) = L^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

Then by formula V we get

$$L\left\{\frac{f(t)}{t}\right\}\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{s}{\left(s^2+a^2\right)}ds = \frac{1}{2}\int_s^\infty \frac{2s}{\left(s^2+a^2\right)^2}ds$$

$$= -\frac{1}{2}\left[\frac{1}{s^2+a^2}\right]_s^\infty$$

$$= -\frac{1}{2}.\frac{1}{\left(s^2+a^2\right)}$$

$$\therefore \quad \frac{f(t)}{t} = \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s^2+a^2\right)}\right\} = \frac{1}{2a}.\sin at$$

Hence $f(t) = L^{-1}\left\{\dfrac{1}{\left(s^2+a^2\right)^2}\right\} = \dfrac{1}{2a}.t\sin at$

**LAPLACE TRANSFORM METHOD TO SOLVE LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS ASSOCIATED WITH INITIAL CONDITIONS :**

Linear differential equations with constant coefficients associated with initial conditions can be easily solved by Laplace transform method.

**Working Procedure :**

**Step - 1 :** Take the Laplace transform of both sides of the differential equation and then put the given initial conditions.

**Step - 2 :** Transpose the terms with minus signs to the right.

**Step - 3 :** Divide by the co-efficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

**Step - 5 :** Resolve this function of s into partial fractions.

**Step – 5 :** Take the inverse Laplace transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

**Example – 22 :**

Solve the following equation by transform method;

$y'' - 3y' + 2y = e^{3t}$, when y (0) = 1 and y′(0) = 0.

**Solution :**

We have, $y'' - 3y' + 2y = e^{3t}$ ...(1)

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} = 3L\{y'\} + 2L\{y\} = L\{e^{3t}\}$$

$$\Rightarrow \quad [s^2\bar{y} - sy(0) - y'(0)] - 3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}$$

Putting y (0) = 1 and y′ (0) = 0, we get

$$s^2\bar{y} - s - 3s\bar{y} + 3 + 2\bar{y} = \frac{1}{s-3}$$

$$\Rightarrow \quad \bar{y}.(s2 - 3s + 2) = \qquad + s - 3 = \frac{1+(s-3)^2}{s-3}$$

$$\Rightarrow \quad \bar{y} = \frac{s^2-6s+10}{(s-3)(s^2-3s+2)}$$

$$\Rightarrow \quad \bar{y} = \frac{s^2-6s+10}{(s-3)(s-1)(s-2)} \qquad \text{...(2)}$$

Let, $\dfrac{s^2-6s+10}{(s-3)(s-1)(s-2)} = \dfrac{A}{s-3} + \dfrac{B}{s-1} + \dfrac{C}{s-2}$

Multiplying both sides of (3) by (s –3) (s – 1) (s – 2), we get

$s^2 - 6s + 10 = A(s-1)(s-2) + B(s-3)(s-2) + C(s-3)(s-1)$ ...(4)

Putting s = 1, B = $\dfrac{5}{2}$

Putting s = 2, C = –2

Putting s = 3, A = $\dfrac{1}{2}$

Substituting the values of A, B, C in (3), we get

$$\bar{y} = \frac{1}{2}.\frac{1}{(s-3)} + \frac{5}{2}.\frac{1}{(s-1)} - 2.\frac{1}{(s-2)}$$

Taking inverse Laplace transform of both sides, we get

$$L^{-1}\{\bar{y}\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - 2L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\therefore \quad y = \frac{1}{2}e^{3t} + \frac{5}{2}e^{t} - 2e^{2t}$$

This is the required solution.

**Example – 23 :**

Solve the following equation by transform method;

$(D^2 + \omega^2)\ y = \cos \omega t$, $t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$

**Solution :**

We have

$(D^2 + \omega^2)\ y = \cos \omega t$

*i.e.*, $y'' + \omega^2 y = \cos \omega t$, given $y(0) = y'(0) = 0$

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} + \omega^2 L\{y\} = L\{\cos \omega t\}$$

$$\Rightarrow \quad s^2 \bar{y} - sy(0) - y'(0) + \omega 2\, \bar{y} = \frac{s}{s^2 + \omega^2}$$

Putting (0) = 0 and y′ (0) = 0 , we get

$$\bar{y} \,.\, (s^2 + \omega^2) = \frac{s}{s^2 + \omega^2}$$

$$\Rightarrow \quad \bar{y} = \frac{s}{\left(s^2 + \omega^2\right)^2} \qquad \text{...(2)}$$

Taking inverse Laplace transform, we get

$$L^{-1}\{\bar{y}\} L^{-1} = \frac{s}{\left(s^2 + \omega^2\right)^2}$$

$$\therefore \quad y = \frac{1}{2\omega} \,.\, t \sin \omega t$$

This is the required solution.

## *Assignment*

**1.** Find the Laplace transforms of the following :

(a) $\left\{ \int_0^t \frac{\sin t}{t} \right\}$ (b) $L\left\{ \int_0^t e^{-t} \cos t \, dt \right\}$

**2.** Find the Laplace Transform of f(t) in each of the following :

(a) $f(t) = \begin{cases} \sin 2t, & \text{when } 0 < t \le \pi \\ 0, & \text{when } t > \pi \end{cases}$ (a) $f(t) = \begin{cases} 1, & \text{when } 0 \le t \le 2 \\ t, & \text{when } t > 2 \end{cases}$

3. Obtain the inverse Laplace transforms ofthe following functions

(a) $\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$ (b) $\dfrac{a(s^2 - 2a^2)}{s^4 + 4a^4}$

(c) $\dfrac{s}{s^2 + 6s + 13}$ (d) $\log \left( \dfrac{1+s}{s} \right)$

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# FOURIER SERIES

**Periodic Functions :**

If the value of each ordinate $f$(t) repeat it self at equal interval in the abscissa, then f(t) is said to be a periodic function.

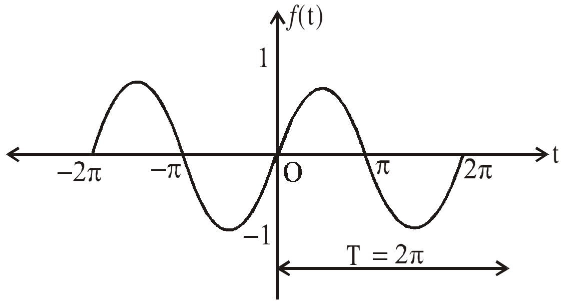
If $f(t) = f(t + T) = f(t + 2T) = .....$, then

T is called period of the function $f(t)$.

For example

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = ......$

So sin $x$ is called a periodic function of period $2\pi$




**Founier Series :**

A series of sines and cosines of an angle and its multiple of the form

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + .... + a_n \cos nx + .....$$

$$+ b_1 \sin x + b_2 \sin 2x + ...... + b_n \sin nx + .....$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called a fouries series, there $a_0$, $a_n$ & $b_n$ are called fourier constants

**Useful Integrals**

The following integrals are usful in Fourier series :

**1.** $\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = 0$ **2.** $\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = 0$

**3.** $\int_{\alpha}^{\alpha+2\pi} \sin^2 nxdx = \pi$ **4.** $\int_{\alpha}^{\alpha+2\pi} \cos^2 nxdx = \pi$

**5.** $\int_{\alpha}^{\alpha+2\pi} \sin nx.\sin mxdx = 0$ **6.** $\int_{\alpha}^{\alpha+2\pi} \cos nx.\cos mxdx = 0$

**7.** $\int_{\alpha}^{\alpha+2\pi} \sin nx.\cos mxdx = 0$ **8.** $\int_{\alpha}^{\alpha+2\pi} \sin nx.\cos nxdx = 0$

**9.** $\int uv = uv_1 - u'v_2 + u''v_3 - ..........$

where $v_1 = \int vdx, v_2 = \int v_1 dx, v_3 = \int v_2 dx \quad u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}$ &

**10.** sin nπ = 0 & cos nπ = (–$x$) where n ∈ I

Let $f(x)$ be represented in the interval (α, α + 2π) by fourier series

$$f(x) = \frac{a_0}{2} + \sum_{x_1=1}^{\infty} a_n \cos nx + \sum_{x_1=1}^{\infty} b_n \sin nx \qquad ...(1)$$

**To find $a_0$ :**

Integrate both sides of equation (1) form $x = \alpha$ to $x = \lambda + 2\pi$ . Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{1}{2}a_1 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{1}{2}a_0(\alpha + 2\pi - \alpha) + 0 + 0 = a_0\pi$$

Hence $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx$

**To find $a_n$ :** Muliply cos $nx$ $a_n$ both sides of equation (1) and integrate from $x$ = 2 to $x$ = 2 + 2R, Then

$$\int_{\pi}^{\pi+2\pi} f(x) \cos nxdx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nxdx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nxdx$$

$$+ \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right) \text{con} nxdx = 0 + \pi a_n + 0$$

$$a_n + \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nxdx$$

**To find $b_n$ :** Multiply sin $nx$ on both sides of equation (1) and intergrate from $x = \pi$ to $x = \pi + 2\pi$ , then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nxdx = \frac{a_0}{2}\int_{2}^{2+\pi} \sin nxdx +$$

$$\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nxdx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nxdx = 0 + 0 + \pi b_n$$

Hence $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$

Making $\alpha = 0$, the interval becomes $0 < x < \pi$ and the formula (1) reduces to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \end{aligned}\right\} \quad ...(ii)$$

$$b_n = \frac{1}{\pi}\int_0^{2R} f(x)\sin nx dx$$

Putting $\alpha = -\pi$, The interval becomes $-\pi < x < \pi$, the formula (I) reduces to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \end{aligned}\right\} \quad ....(iii)$$

**Euler's Formula :**

The fourier series for the function $f(x)$ in the interval $\pi < x < \pi + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$$

The value of $a_0$, $a_n$ & $b_n$ are known.

Euler's formula.

**Example – 1 :**

Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expansion of $f(x)$. Hence that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + .......$$

**Solution :**

Let $x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + .....$

$+b_1 \sin x + b_2 \sin 2x + ......$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx dx$$

$$= \frac{1}{\pi}\left[\left(x+x^2\right)\frac{\sin nx}{x} - \left(2x+1\right)\frac{\left(-\cos nx\right)}{x^2} + 2.\left(\frac{-\sin nx}{x^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{x^2} - \left(-2\pi+1\right)\frac{\cos(-n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[4\pi\cdot\frac{\cos n\pi}{x^2}\right] = \frac{4.(-1)^x}{x^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\cdot\frac{-\cos nx}{x} - (2x+1)\cdot\left(\frac{-\sin nx}{x^2}\right) + 2\frac{\cos nx}{x^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\cdot\frac{\cos nx}{x} - 2\cdot\left(\frac{\cos nx}{x^3}\right) + \left(-\pi+\pi^2\right)\frac{\cos nx}{x} - \frac{2\cos n\pi}{x^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{\pi}.\cos n\pi\right] = \frac{-2}{n}.(-1)^n$$

Substituting the values of $a_0$, $a_n$ & $b_n$ is equation (1)

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x \frac{-1}{3^2}\cos 3x + ....\right.$$

$$\left.-2\left[-\sin x + \frac{1}{2}\sin 2\pi - \frac{1}{3}\sin 3x + .....\right]\right. \quad ...(2)$$

Put $x = \pi$ in equation (2)

$$\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....\right] \qquad ...(3)$$

Put $x = -\pi$ in equation (2)

$$-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+......\right] \qquad ...(4)$$

Adding equation (3) & (4)

$$2\pi^2=\frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.......\right]$$

$$\frac{4\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.......\right]$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.......=\sum_{n=1}^{\infty}\frac{1}{x^2}$$

## Dirchelet's Condition :

Any function $f(x)$ can be developed as a fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

where $a_0$, $a_n$, $b_n$ are constants provided.

(i) $f(x)$ is periodic, single valued and finite

(ii) $f(x)$ has a finite no.of discontinuities in any one period

(iii) $f(x)$ has at most a finite no.of maxima and minima.

**Discontinuous Functions :** At a point of discontinuty, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At a point of discontinuty, $x = c$

$$f(x)=\frac{1}{2}[f(c-0)+f(c+0)]$$

**Example – 2 :**

Find the fourier series expansion for

$f(x)$, if $f(x)=\begin{cases}-\pi, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\frac{\pi^2}{8}$

**Solution :**

Let $f(x)=\frac{a_0}{z}+\sum_{x=1}^{\infty}a_n\cos nx+\sum_{x=1}^{\infty}b_n\sin nx$ ...(1)

then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} xdx\right]$

$$= \frac{1}{\pi}\left[-\pi\{x\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nxdx + \int_{0}^{\pi} x\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\cdot\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{x\cdot\frac{\sin nx}{x} + \frac{\cos nx}{x^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{x^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[\cos n\pi - 1\right]$$

$$a_1 = \frac{-2}{\pi.1^2},\ a_2 = 0,\ a_3 = \frac{-2}{\pi.3^2},\ a_4 = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi).\sin nxdx + \int_{0}^{\pi} x\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi.\cos nx}{x}\right\}_{-\pi}^{0} + \left\{\frac{-x.\cos nx}{x} + \frac{\sin nx}{x^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{x}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{x}(1 - 2\cos n\pi)$$

$$\therefore\ b_1 = 3,\ b_2 = \frac{-1}{2},\ b_3 = 1,\ b_4 = \frac{-1}{4}$$

Substituting the values of a's and b's in equation (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ....\right]$$

$$+3\sin x - \frac{\sin 2x}{2} + 3\frac{\sin 3x}{3} - ....\quad ...(ii)$$

Putting $x = 0$ in equation (ii)

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + .....\infty\right)\quad ...(iii)$$

Now $f(x)$ is discontinuous at $x = 0$

But $f(0-0) = -\pi$ and $f(0+0) = 0$

$$\therefore\ f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{-\pi}{2}$$

From equation (iii)

$$-\frac{\pi}{2}=\frac{-\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right]$$

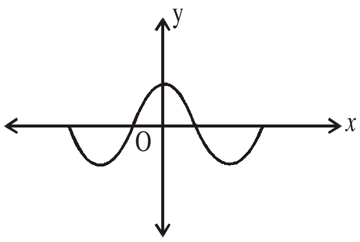
or $$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.........$$

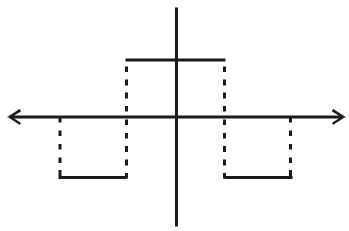
**Even Function :** A function $f(x)$ is said to be even (or symmetric) function if $f(-x) = f(x)$

**Ex.** (i) $x^2$ , $x^4$, $x^6$,....... even powers. of $x$

(ii) $\cos x$ , $\sec x$ etc.

The graph of such a function is symmetrical with respect to y-axis. Here y-axis is a mirror for the reflection of the curve



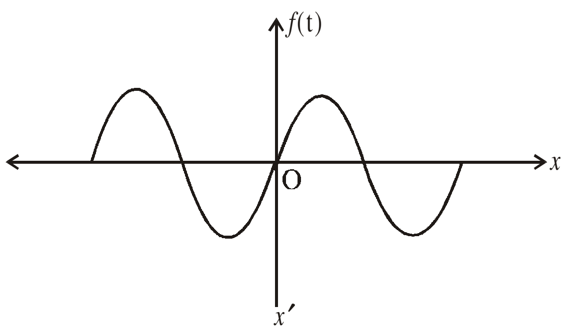
The area under such a curve from $-\pi$ to $\pi$ is double the area from 0 to $\pi$.

$$\therefore \int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{x} f(x)dx$$

**Odd function :** A function $f(x)$ is called odd (skew symmetric) function if $f(-x) = -f(x)$

**Ex.** (i) $x^3$, $x^5$, $x^7$,...... odd powers if $x$

(ii) $\sin x$, $\operatorname{cose} x$, $\tan x$ etc




Here the area under the curve from $-\pi$ to $\pi$ is zero

i.e., $\int_{-\pi}^{\pi} f(x)dx = 0$

**Expansion of an Even Function :**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nxdx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$$

As $f(x)$ and connx both are even, the product of $f(x)$.cosnx is also even.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$$

As sin nx is an odd function. The product of an even function with odd function is odd. therefor we need not calculate $b_n$.

The serives of an even function contain cosine terms only.

**Expansion of an odd Function :**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\, dx = 0 \quad (\because f(x)\text{ . cosnx is odd function})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\, dx$$

$(\because f(x).\sin nx$ is even function)

The series of an odd function contain sine terms only.

**Example – 3 :**

Obtain a founier expansion of for $f(x) = x^3$. in $-\pi < x < \pi$

**Solution :**

$f(x) = x^3$ is an odd function.

$\therefore a_0 = 0$ and $a_n = 0$

$$b_x = \frac{2}{\pi}\int_0^{\pi} \phi(x).\sin nxdx = \frac{2}{\pi}\int_0^{\pi} x^3.\sin nxdx$$

$$= \frac{2}{\pi}\left[x^3.\frac{-\cos nx}{x} - 3x^2.\frac{-\sin nx}{x^2} + bx.\frac{\cos nx}{x^3} - 6.\frac{\sin nx}{x^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x^3.\frac{\cos nx}{x} + 6x.\frac{\cos nx}{x^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi^3.\frac{\cos n\pi}{x} + 6.\pi.\frac{\cos n\pi}{x^3}\right]$$

$$= 2.(-1)^x \left[ \frac{-\pi^2}{x} + \frac{6}{x^3} \right]$$

$$\therefore\ x^3 = 2 \left[ \left( \frac{-\pi^2}{1} + \frac{6}{1^3} \right) \sin x + \left( \frac{-\pi^2}{2} + \frac{6}{2^3} \right) \sin 2x - \left( \frac{-\pi^2}{3} + \frac{6}{3^3} \right) \sin 3x + \ldots\ldots \right.$$

**Half Range Series :**

To obtain a Fourier expansion of a function $f(x)$ for the range $(0, \pi)$ which is half the period of the fourier series. As it is immateriad what ever the function many be outside the range $0 < x < \pi$, we extend the function to cover the range $-\pi < x < \pi$. So that the new function may be even or odd. The fourier expansion of such function of half the period consists sine or cosire term only.

**Sine Series :**

If it is required to expand $f(x)$ as a sine aeries in $0 < x < \pi$ me extend the function to the range $-\pi < x < \pi$, so that if will be an odd function.

The desired half-range sin series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

**Cosine Series :**

If its is required to expand $f(x)$ as a cosine series in $0 < x < \pi$, We extend the function to the range $-\pi < x < \pi$, so that if will be an even function.

The desired half – range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

**Example – 4 :**

Find the half - range sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$

**Solution :**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx\, dx$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\pi - n\cos n\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\cdot\frac{n}{a^2+n^2}\left[-(-1)^n e^{a\pi} + 1\right]$$

$$= \frac{2n}{(a^2+n^2)\pi}\left[1-(-1)^n e^{a\pi}\right]$$

$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1)\pi}, b_2 = \frac{2.2(1-e^{a\pi})}{(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x + .....\right]$$

## *Assignment*

**1.** Find a fourier series to represent $f(x) = \pi - x, 0 < x < 2\pi$

**2.** Find a fourier series to represent the function

$f(x) = e^x$, for $-\pi < x < \pi$

**3.** Find the fourier series of the function $f(x) = \begin{cases} -1, & for\ -\pi < x < -\frac{\pi}{2} \\ 0, & for\ -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1, & for\ \frac{\pi}{2} < x < \pi \end{cases}$

**4.** Represent the following function by a fourier sine series $f(t) = \begin{cases} t, & 0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < t \le \pi \end{cases}$

**5.** Find the fourier cosine series for the function $f(x) = \begin{cases} 1, & for\ 0 < x < \frac{\pi}{2} \\ 0, & for\ \frac{\pi}{2} < x < \pi \end{cases}$

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# FINITE DIFFERENCE AND INTERPOLATION

**Finite Difference :**

Suppose we are given the following values of y = $f(x)$ for a set of values $fx$ :

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $x_0$ | $x_1$ | $x_2$ | ........ | $x_n$ |
| y | $y_0$ | $y_1$ | $y_2$ | ........ | $y_n$ |

The interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. While the process of computing the value of the function outside the given range is calle extrapolation.

Suppose that the function y = $f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h$, $x_0 + 2h$, ....... $x_0 + nh$ giving y = $y_0, y_1$ ....... $y_n$. To determine the values of $f(x)$ for some intermediate values of $x$, the following two types of difference are found useful.

Forward difference - The differences

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Similarly

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^3 y_1 = \Delta^2 y_1 - \Delta^2 y_1$$

$$\Delta^n y_0 = \Delta^{n-1} y_1 - \Delta^{n-1} y_0$$

Forward difference table

| Value of $x$ | Value of y | 1st diff. | 2nd diff. | 3rd diff. | 4th diff. | 5th diff. |
|---|---|---|---|---|---|---|
| $x_0$ | $y_0$ | $\Delta y_0$ | | | | |
| $x_0 + h$ | $y_1$ | $\Delta y_1$ | $\Delta^2 y_0$ | | | |
| $x_0 + 2h$ | $y_2$ | $\Delta y_2$ | $\Delta^2 y_1$ | $\Delta^3 y_0$ | | |
| $x_0 + 3h$ | $y_3$ | $\Delta y_3$ | $\Delta^2 y_2$ | $\Delta^3 y_1$ | $\Delta^4 y_0$ | |
| $x_0 + 4h$ | $y_4$ | $\Delta y_4$ | $\Delta^2 y_3$ | $\Delta^3 y_2$ | $\Delta^4 y_1$ | $\Delta^5 y_0$ |
| $x_0 + 5h$ | $y_5$ | | | | | |

Backward difference

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_1 = y_1 - y_0$$

:

:

:

$$\nabla^n y_n = \nabla^{n-1} y_n - \nabla^{n-1} y_{n-1}$$

Backward difference table

| Value of $x$ | Value of y | 1st diff. | 2nd diff. | 3rd diff. | 4th diff. | 5th diff. |
|---|---|---|---|---|---|---|
| $x_0$ | $y_0$ | $\Delta y_0$ | | | | |
| $x_0 + h$ | $y_1$ | $\Delta y_1$ | $\Delta^2 y_0$ | | | |
| $x_0 + 2h$ | $y_2$ | $\Delta y_2$ | $\Delta^2 y_1$ | $\Delta^3 y_0$ | | |
| $x_0 + 3h$ | $y_3$ | $\Delta y_3$ | $\Delta^2 y_2$ | $\Delta^3 y_1$ | $\Delta^4 y_0$ | |
| $x_0 + 4h$ | $y_4$ | $\Delta y_4$ | $\Delta^2 y_3$ | $\Delta^3 y_2$ | $\Delta^4 y_1$ | $\Delta^5 y_0$ |
| $x_0 + 5h$ | $y_5$ | | | | | |

**Differences of a polynomial**

We know that the expression of the form $f(x) = a_0 x^n + a_2 x^{n-1} + ..... + a_{n-1} x + a_n$ where a's are constant ($a_0 \neq 0$) and n is a positive integer is called a polynomial in $x$ of degree n.

**Theorem :**

The 1st difference is a polynomial of degree n is of degree n – 1, the 2nd difference is of degree n – 2, and the nth difference is constant. While higher difference are equal to zero.

The converse of the theorem is alos true which stated that if nth difference of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n.

**Example – 1 :**

Form the successive forward differences of $ax^3$, the interval being h.

**Solution :**

Here $y = f(x) = ax^3$

We know that

$$\Delta y_0 = y_1 - y_0 = f(x + h) - f(x)$$

$$\therefore \Delta (ax^3) = a(x - h)^3 - ax^3$$

$$= a(x^3 + 3x^2h + 3gh^2 + h^3) - ax^3$$

$$= a(3x^2h + 3xh^2 + h^3)$$

Again, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$$\therefore \Delta^2(ax^3) = a\{3(x + h)^2 h + 3(x + h)h^2 + h^3\} - a(3x^2h + 3xh^2 + h^3)$$

$$= a\{3x^2h + 6xh^2 + 3h^3 + 3xh^2 + 3h^3 + h^3 - 3x^2h - 3xh^2 - h^3\} = a\{6xh^2 + 6h^3\}$$

$\Delta^3 y_0 = \Delta^2 y_1 = \Delta^2 y_0$

$\therefore \Delta^3 (ax^3) = a\{6(x + h) h^2 + 6h^3\} - a\{6xh^2 + 6h^3\}$

$= a\{6xh^2 + 6h^3 + 6h^3 - 6xh^2 - 6h^3\} = 6ah^3 = \text{Constant}$

and $\Delta^4 y_0 = \Delta^3 y_1 = \Delta^3 y_0$

$\therefore \Delta^4(ax^3) = 6ah^3 - 6ah^3 = 0$

Here it shows that the the third differences of a polynomial of third degree is constant & the higher difference & are zero.

Factorial Notation

A product of the form $x(x - 1)(x - 2)$ ........ $(x - r + 1)$ is devoted by $[x]^r$ and is called a factorial.

In particular

$[x] = x$, $[x]2 = x(x - 1)$

$[x]^3 = x(x - 1)(x - 2)$

$[x]^n = x(x - 1)(x - 2) \ldots\ldots (x - n + 1)$

which is called a factorial polynomial or function.

The factorial notation is of special utility in the theory of first differences. It helps in finding the successive differences of a polynomials directly by simply rule of differentiation.

The result of differentiating $[x]^r$ is similar to that of differential $x^r$.

**Example – 2 :**

Estimate the missing term in the following table :

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 9 | 8 | 1 |

**Solution :**

Let the missing term by $y_1$. The following is the difference table.

| $x$ | y | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| 1 | 3 | 2 | 4 | $y_3 - 19$ | |
| 2 | 9 | 6 | $y_3 - 15$ | | $124 - 4y_3$ |
| 3 | $y_3$ | $y_3 - 9$ | $81 - 2y_3 + 9$ | $105 - 3y_3$ | |
| 4 | 81 | $81 - y_3$ | | | |

As only four entries $y_0$, $y_1$, $y_2$, $y_4$ are given, the function y can be represented by a third degree polynomial, here 4th order difference becomes zero, i.e.,

$124 - 4y_3 = 0$

$\Rightarrow y_3 = 31$

Hence the missing term is 31.

**Example – 3 :**

Estimate the missing term in the following table :

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| y | 5 | 11 | 22 | 40 | -- | 140 | -- |

**Solution :**

Let the missing term by $y_4$ & $y_6$. The following is the difference table.

| $x$ : | y: | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|
| 0 | 5 | | | | | |
| 1 | 11 | 6 | 5 | | | |
| 2 | 22 | 11 | 7 | 2 | $y_4 - 67$ | |
| 3 | 40 | 18 | $y_4 - 40$ | $y_4 - 58$ | $303 - 4y_4$ | $370 - 5y_4$ |
| 4 | $y_4$ | $y_4 - 40$ | $180 - 3y_4$ | $238 - 3y_4$ | $y_6 + 6y_4 -$ | $y_6 + 10y_4 - 1001$ |
| 5 | 140 | $140 - y_4$ | $y_6 + y_4 -$ | $y_6 + 3y_4 - 460$ | 698 | |
| 6 | $y_6$ | $y_6 - 140$ | 280 | | | |

As only four entries $y_0$, $y_1$, $y_2$, $y_4$, $y_5$ are given, the function y can be represented by a $4^{th}$ degree polynomial & hence $5^{th}$ difference becomes zero, i.e.,

$370 - 5y_4 = 0$ and $y_6 + 10y_4 - 1001 = 0$

Solving these, we get

$y_4 = 74$ and $y_6 = 261$

**Newton's Forward interpolation formula for equal intervals**

Let the function y = $f(x)$ takes the values $y_0$, $y_1$, $y_2$.......... corresponding to the values $x_0$, $x_1$ + h, $x_0$ + 2h of $x$.

$$f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + ......$$

**Obs.** This formula is used for interpolating the values of y near the beginning of a set of tabulated values and exterpolating values of y a little backward (i.e. to the left) of $y_0$.

**Newton's backward interpolation formula for equal intervals**

Let the function y = $f(x)$ takes the values $y_0$, $y_1$, $y_2$........ corresponding to the values $x_0$, $x_1$ + h, $x_0$ + 2h of $x$.

$$f(x_n + nh) = y_n + n\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + ......$$

**Obs.** This formula is used for interpolating the values of y near the beginning of a set of tabulated values and exterpolating values of y a little backward (i.e. to the right) of $y_n$.

**Example – 4 :**

The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

| $x$ = height | 100 | 150 | 200 | 250 | 300 | 350 | 400 |
|---|---|---|---|---|---|---|---|
| $y$ = distance | 10.63 | 13.03 | 15.04 | 16.81 | 18.42 | 19.90 | 21.27 |

Find the values of y when (i) $x = 218$ ft.

**Solution :**

The difference table is as under :

| ***x*** | **y** | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|
| 100 | 10.63 | | | | |
| | | 2.40 | | | |
| 150 | 13.03 | | – 0.39 | | |
| | | 2.01 | | 0.15 | |
| 200 | **15.04** | | – 0.24 | | – 0.07 |
| | | **1.77** | | 0.08 | |
| 250 | 16.81 | | **– 0.16** | | – 0.05 |
| | | 1.61 | | **0.03** | |
| 300 | 18.42 | | – 0.13 | | **– 0.01** |
| | | 1.48 | | **0.02** | |
| 350 | 19.90 | | **– 0.11** | | |
| | | **1.37** | | | |
| 400 | **21.27** | | | | |

(i) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since x = 218 and h = 50, $\therefore\ n = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + n\Delta y_0 + \frac{n(n-1)}{1.2}\Delta^2 y_0 \frac{n(n-1)(n-2)}{1.2.3}\Delta^3 y_0 + .....$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36 + (-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + ....$$

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ taking $x_n = 400, n = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward differences

$Y_n = 21.27$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ etc.

∴ Newton's backward formula gives

$$y_{410} = y_{400} + n\nabla y_{200} + \frac{n(n+1)}{2}\nabla^2 y_{400} + \frac{n(n+1)\,(n+2)}{1.2.3}\nabla^3 y_{400} + ....$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + .... = 21.53 \text{ nautical miles.}$$

**Example – 5 :**

Find the number of men getting wages between Rs. 10 and 15 from the following data :

| Wages in Rs. | 0 – 10 | 10 – 20 | 20 – 30 | 30 – 40 |
|---|---|---|---|---|
| Frequency | 9 | 30 | 35 | 42 |

**Solution :**

First we prepare the cumulative frequency table, as follows :

| Wages less than ($x$) | 10 | 20 | 30 | 40 |
|---|---|---|---|---|
| No.of men (y) | 9 | 39 | 74 | 116 |

Now the difference table is

| $x$ | y | $\Delta$ | $\Delta^2$ | $\Delta^3$ |
|---|---|---|---|---|
| 10 | 9 | | | |
| | | 30 | | |
| 20 | 39 | | 5 | |
| | | 35 | | 2 |
| 30 | 74 | | 7 | |
| | | 42 | | |
| 40 | 116 | | | |

We shall find $y_{15}$ i.e. number of men getting wages less than 15.

Taking $x_0 = 10$, $x = 15$, we have

$$n = \frac{x - x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

∴ using Newton's forward interpolation formula, we get

$$y_{15} = y_{10} + n\Delta y_{10} + \frac{n(n-1)}{2!}\Delta^2 y_{10} + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_{10}$$

$$= 9 + (0.5)\times 30 + \frac{(0.5)\,(0.5-1)}{2}\times 5 + \frac{(0.5)(0.5-1)(0.5-2)}{6}\times 2$$

$$= 9 + 15 - 0.625 + 0.125 = 23.5 = 24 \text{ approx.}$$

Number of men getting wages between Rs. 10 and 15 = 24 – 10 = 5 approx.

**Example – 6 :**

Find the cubic polynomial which takes the following values :

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 1 | 2 | 1 | 10 |

**Solution** :

The difference table is

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| | | 1 | | |
| 1 | 2 | | –2 | |
| | | –1 | | 12 |
| 2 | 1 | | 10 | |
| | | 9 | | |
| 3 | 10 | | | |

We take $x_0 = 0$ and $p = \frac{x-0}{h} = x$

$\therefore$ using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)\,(x-2)}{6}(12)$$

$= 2x^3 + 7x^2 + 6x + 1$, which is the required polynomial.

To compute f(4), we take $x_n = 3$, $x = 4$ so that $p = \frac{x - x_n}{h} = 1$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + n\nabla f(3) + \frac{n(n+1)}{1.2}\nabla^2 f(3) + \frac{n(n+1)(n+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 + 41$$

which is the same value are that obtained by substituing $x = 4$ in the cubic polynomial above.

**Obs.** The above example shows that if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.

**Lagrange's Interpolation formula for unequal intervals :**

$$f(x) = \frac{(x-x_1)\,(x-x_2).....(x-x_n)}{(x-x_1)\,(x_0-x_2).....(x_0-x_n)}$$

$$y_0 = \frac{(x-x_1)\,(x-x_2).....(x-x_n)}{(x_1-x_0)\,(x_1-x_2).....(x_1-x_n)}$$

$$y_1 + ....... \frac{(x-x_0)\,(x-x_1).....(x-x_{n-1})}{(x_n-x_0)\,(x_n-x_2).....(x_n-x_{n-1})}$$

**Lagrange's Method for unequally spaced values of *x* :**

$$x = \frac{(y-y_1)\,(y-y_2)........(y-y_n)}{(y_0-y_1)\,(y_0-y_2)........(y_0-y_n)} + \frac{(y-y_0)\,(y-y_2)........(y-y_n)}{(y_1-y_0)\,(y_1-y_2)........(y_1-y_n)}$$

$$+....+ \frac{(y-y_0)\,(y-y_1)........(y-y_{n-1})}{(y_n-y_0)\,(y_n-y_1)........(y_n-y_{n-1})}$$

**Example – 7 :**

Use lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of $x$ & y are given.

**Solution :**

| | | | | |
|---|---|---|---|---|
| Here | $x_0 = 5$ | $x_1 = 6$ | $x_2 = 9$ | $x_3 = 11$ |
| and | $y_0 = 12$ | $y_1 = 13$ | $y_2 = 14$ | $y_3 = 16$ |

Putting $x = 10$ and substituting the above value in Lagrange's formula, we get :

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}.y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}.y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.y_3$$

$$f(10) = \frac{(10-6)\,(10-9)\,(10-11)}{(5-6)\,(5-9)\,(5-11)} \times 12 + \frac{(10-5)\,(10-9)\,(10-11)}{(6-5)\,(6-9)\,(6-11)} \times 13$$

$$+ \frac{(10-5)\,(10-6)\,(10-11)}{(9-5)\,(9-6)\,(9-11)} \times 14 + \frac{(10-5)\,(10-6)\,(10-9)}{(10-5)\,(11-6)\,(11-9)} \times 16$$

$$= \frac{4\times1\times(-1)}{(-1)\times(-4)\times(-6)} \times 12 + \frac{5\times1\times(-1)}{1\times(-3)\times(-5)} \times 13$$

$$+ \frac{5\times4\times(-1)}{4\times3\times(-2)} \times 14 + \frac{5\times4\times1}{6\times5\times2} \times 16 \; \frac{-48}{-24} + \frac{-65}{-15} + \frac{-280}{-24} + \frac{320}{60}$$

$= 2 - 4.33 + 11.66 + 5.33 = 14.66$

**Example – 8 :**

Apply lagrange's method to find the value of $x$ when $f(x) = 15$ from the given data.

| $x$ | 5 | 6 | 9 | 11 |
|---|---|---|---|---|
| $f(x)$ | 12 | 13 | 14 | 16 |

**Solution :**

Here

$x_0 = 5,\quad x_1 = 6,\quad x_2 = 9,\quad x_3 = 11$

$y_0 = 12,\quad y_1 = 13,\quad y_2 = 14,\quad y_3 = 16$

Taking y = 15 and using the above results in Lagrange's inverse interpolation formula.

$$x = f(x) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} . x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} . x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} . x_3$$

$$= \frac{(15-13)1(15-14)\ (15-16)}{(13-13)\ (12-14)\ (12-16)} \times 5 + \frac{(15-12)1(15-14)\ (15-16)}{(13-12)\ (13-14)\ (13-16)} \times 6$$

$$+ \frac{(15-12)1(15-13)\ (15-16)}{(14-12)\ (14-13)\ (14-16)} \times 9 + \frac{(15-12)1(15-13)\ (15-14)}{(16-12)\ (16-13)\ (16-16)} \times 11$$

$$= \frac{2\times1\times(-1)}{(-1)\times(-2)\times(-4)} \times 5 + \frac{3\times1\times(-1)}{1\times(-1)\times(-3)} \times 6 + \frac{3\times2\times(-1)}{2\times1\times(-2)} \times 9 + \frac{3\times2\times1}{4\times3\times2} \times 11$$

$$= \frac{5}{4} - 6 + \frac{27}{2} + \frac{11}{4} = 1.25 - 6 + 13.5 + 2.75 = 17.5 - 6 = 11.5$$

## *Assignment*

1. Find a cubic polynomial which takes the following values

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 1 | 2 | 1 | 10 |

2. Given the values

| $x$ | 5 | 7 | 11 | 13 | 17 |
|---|---|---|---|---|---|
| $y$ | 150 | 392 | 1452 | 2366 | 5202 |

Evaluate $y_9$ using Lagrange's formula.

3. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192 and sin 60° = 0.8660. Find sin 52° using Newton's forward interpolation formula.

❒ ❒ ❒

# NUMERICAL SOLUTION OF EQUATION

**1.** An expression of the form

$f(x) = a_0x^n + a_1x^{n-1} + .......+ a_{n-1}x + a_n$

where $a_0, a_1, a_2, ....... a_n \neq 0$ are constant and n is a positive integer is called a polynomial in $x$ of degree n.

**2.** The polynomial $f(x) = 0$

For example (1) $2x^2 + x^2 - 13x + 6 = 0$

(2) $x^3 - 4x + 9 = 0$

are called algebraic equation.

**3.** Transcendental equation -

If $f(x)$ is a functions other than algebraic function such as trigonometric, logarithmic, exponential etc. then $f(x)$ is called transcendental function.

**4.** Root of an equation -

The value of $x$ which satisfied $f(x) = 0$ is called the root of the equation.

Geometrically a root of the equation $f(x) = 0$ & y = 0 is the value of $x$ where the graph meet the y-axis.

**5.** Solution of an equation -

The process of finding a root of an equation is known as the solution of an equation.

**6.** Different methods to solve the equations.

(a) Analytical method

(b) Graphical method

(c) Numerical method

**7.** Limitation of analytical method

This methods produce very exact and accurate results. But it fails in many cases such as it fails to find roots of transcendental equation.

**8.** Limitation of graphical method -

This methods are simple but these methods produce result to a low degree accuracy.

**9.** Advantages of Numerical method –

This methods are often of a repetitive nature. These consist in repeated execution of the same process. Where each step the result of proceeding step is used. This is known as iteration process and is repeated till the result is obtained to a desired degree of accuracy.

The followings are some Numerical methods to find root of algebraic and transcendental equation –

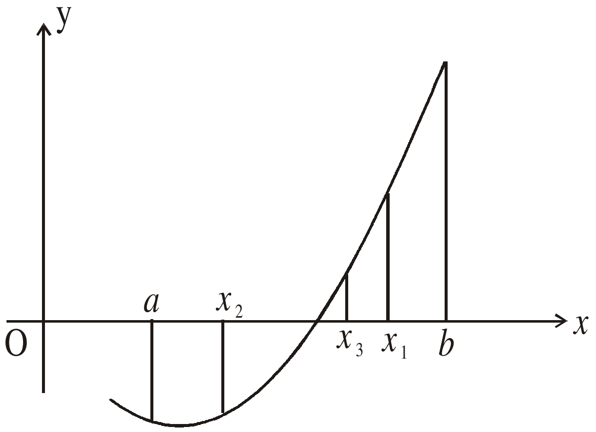
(1) Bisection method

(2) Newton – Raphson method

**Bisection method :**

This method consists of locating a root of the equation $f(x) = 0$ between $a$ and b. If $f(x)$ is continuous between $a$ and b, and $f(a)$ and $f(\text{b})$ are of opposite signs then there is a root between $a$ and b. from the graph $f(a)$ is negative and $f(\text{b})$ is positive then there is a root lies between $a$ and b. The first approximation to the root is

$$x_1 = \frac{1}{2}(a+b)$$

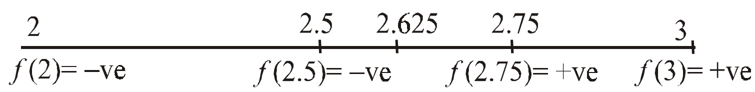



if $f(x) = 0$, then $x_1$ is the root of equation $f(x) = 0$. Otherwise the root lies between $a$ and $x_1$ or $x_1$ and b according to $f(x_1)$ is positive or negative. Then we bisect the interval and continue the process until the root is found to desired accuracy.

In the fig.1 $f(x_1)$ is +ve, so the root lies between $a$ and $x_1$. Then the 2[nd] approximation to the root is $x_2 = \frac{1}{2}(a+x_1)$ . If $f(x_1)$ is –ve, the root lies between $x_1$ and $x_2$. So the third approximation to the root is $x_3 = \frac{1}{2}(x_1+x_2)$ and so on.

**Example – 1 :**

**(a)** Find a root of the equation $x^3 - 4x - 9 = 0$ using the bisection method correct to three decimal places.




**Solution :**

Let $f(x) = x^3 - 4x - 9$

$f(2) = (2)^3 - 4(2) - 9 = -9$ (–ve)

$f(3) = (3)^3 - 4(3) - 9 = 6$ (+ve)

a root lies between 2 and 3.

First approximate to the root is

$$x_1 = \frac{1}{2}(2+3) = 2.5$$

$$f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 \ (-ve)$$

the root lies between $x_1$ and 3. The second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = \frac{1}{2}(2.5+3) = 2.75$$

$$f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \ (+ve)$$

the root lies between 2.5 and 2.75

So $x_3 = \frac{1}{2}(2.5 + 2.75) = 2.625$

$f(2.625) = (2.625)^3 - 4\,(2.625) - 9 = -1.4121 \ (-ve)$

the root lies between 2.625 and 2.75.

$x_4 = \frac{1}{2}\ (2.625 + 2.75) = 2.6875$

Repeating this process, the successive approximation are

$x_5 = 2.71875$, $x_6 = 2.70313$, $x_7 = 2.71094$, $x_8 = 2.70703$, $x_9 = 2.70508$,
$x_{10} = 2.70605$, $x_{11} = 2.70654$, $x_{12} = 2.70642$. Hence the root is 2.7064.

**Example – 2 :**

Find the root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3, using bisection method taking 2 stages.

**Solution :**

Let $f(x) = x \log_{10}x - 1.2 = 2 \times \log_{10} 2 - 1.2$

$= 2 \times .3010 - 1.2 = -0.5979 \ (-ve)$

$f(3) = 3 \times \log_{10}x - 1.2 = 0.2314 \ (+ve)$

the root lies between 2 and 3

$x_1 = \frac{1}{2}(2+3) = 2.5$

$f(2.5) = 2.5\,(\log_{10}2.5) - 1.2 = -0.205 \ (-ve)$

the root lies between 2.5 and 3

$$x_2 = \frac{1}{2}(2.5 + 3) = 2.75$$

Hence the root is 2.75.

**Example – 3 :**

By using the bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.

**Solution.**

Let $f(x) = x \sin x - 1$. We know that V = 57.3°

Since $f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

∴ first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25)\sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

∴ a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1+1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

∴ a root lies between 1 and $x_2 - 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1+1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$

and $f(x_2) > 0$, i.e. now the root lies between $x_4 = 1.0625$ and $x_2 = 1.125$.

∴ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

∴ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

∴ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

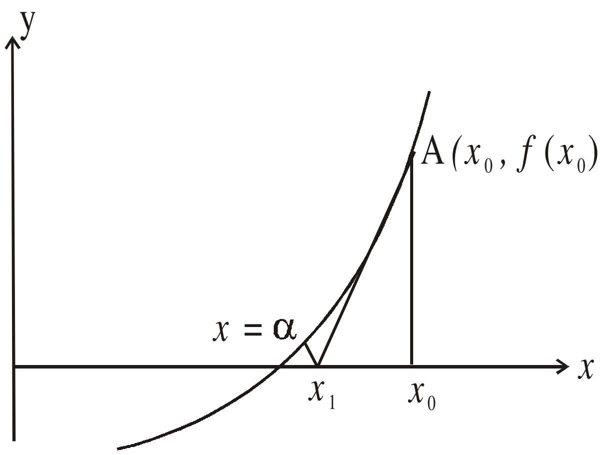
Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

∴ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

**Newton-Raphson Method**




In this method, instead of taking two initial rough approximations to the root $x = \alpha$ as in the previous two methods, a single rough approximation $x_0$ to the root is taken. Then we use the following formula, known as Newton-Raphson formula or Newton iteration formula, to get the successive approximations.

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} \qquad \text{......(1)}$$

Putting n = 0, 1, 2, .................etc. in the above formula (1), we get the first, second, third approximations as follows.

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

This method is useful in cases of large values of $f^1(x)$ i.e., when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

The process of finding successive approximations to the root (i.e., $x_1$, $x_2$, $x_3$ etc.) may be continued till the root is found to desired degree of accuracy.

**Example – 4 :**

Find by Newton's method, a root of the equation $x^3 - 3x + 1 = 0$ correct to 3 decimal places.

**Solution :**

Let $\quad f(x) = x^3 - 3x + 1$

$f(1) - 1 - 3 + 1 = -1$

$f(2) = 2^3 - 3.2 + 1 = 8 - 6 + 1 = 3$

$\Rightarrow$ The root of $f(x)$ lies between 1 & 2

Let $x_0$ = 1.5, Also $f_1(x) = 3x^2 - 3$

Newton's formula gives

$$x_{n+1} = x_n = \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{(x_n^3 - 3x_n + 1)}{2x_n^2 - 3}$$

$$= \frac{3x_n^3 - 3x_n - 3x_n^3 + 3x_n - 1}{3x_n^2 - 3} = \frac{2x_n^3 - 1}{3x_n^2 - 3} \qquad ...(1)$$

Putting n = 0 in (i), the first approximation $x_1$ is given by

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3} = \frac{2\times(1.5)^3 - 1}{3\times(1.5)^2 - 3} = \frac{2\times3.375 - 1}{3\times2.25 - 3} = \frac{5.75}{3.75} = 1.533$$

Putting n = 1 in (i), the second approximation $x_2$ is given by

$$x_2 = \frac{2x_1^3 - 1}{3x_4^2 - 3} = \frac{2\times(1.533)^3 - 1}{3\times(1.533)^2 - 3} = \frac{2\times3.6026 - 1}{3\times2.35 - 3} = \frac{6.2052}{4.05} = 1.532$$

**Example – 5 :**

Find the Newton's method, the real root of the equation $3x = \cos x + 1$

**Solution :**

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 + +\text{ve}$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f(x) = 3 + \sin x$

∴ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad ...(i)$$

Putting n = 0, the first approximation $x_1$ is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3\sin\,(0.6)}$$

$$= \frac{0.6\times0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting n = 1 in (i), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin\,(0.6071)}$$

$$= \frac{0.6071\times0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071$$ Clearly, $x_1 = x_2$.

Hence the desired root is 0.6071 correct to four decimal places.

## *Assignment*

1. Find a root of the following equations, using the bisection method correct to three decimal places.

   (a) $x^3 - x - 11 = 0$

   (b) $x^4 - x - 10 = 0$

2. Find by Newton-Raphson method, a root of the following equations correct to 3 decimal places.

   (a) $x^3 - 3x + 1 = 0$

   (b) $3x^3 - 9x^2 + 8 = 0$

3. Using Newton-Raphson method to evaluate the following

   (a) $\frac{1}{32}$ (b) $\sqrt{41}$

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